RESOLVABILITY OF GRAPHS BASED ON REPRESENTATIONS AND MULTIREPRESENTATIONS

SUPACHOKE ISARIYAPALAKUL

Graduate School Srinakharinwirot University

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RESOLVABILITY OF GRAPHS BASED ON REPRESENTATIONS AND MULTIREPRESENTATIONS

SUPACHOKE ISARIYAPALAKUL

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BY

SUPACHOKE ISARIYAPALAKUL

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(Assoc. Prof. Dr. Chatchai Ekpanyaskul, MD.)

Dean of Graduate School

ORAL DEFENSE COMMITTEE

.... Major-advisor	.... Chair
(Assoc. Prof.Varanoot Khemmani, Ph.D.)	(Chariya Uiyyasathian)
	.... Committee
	(SERMSRI THAITHAE)

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Let G be a connected graph and let $W=\{w_1,w_2,...,w_k\}$ be an ordered set of vertices of G. For the vertex v of G, the representation of v with respect to W is the k-vector $r(v|W)=(d(v,w_1),d(v,w_2),...,d(v,w_k))$, where $d(v,w_i)$ for $i=1,2,...,k$ is the distance between v and w_i in G. An ordered set W is a connected local resolving set of G if the representations of every two adjacent vertices of G with respect to W are distinct and the induced subgraph $\langle W \rangle$ of G is connected. A connected local resolving set of G with minimum cardinality is a minimum connected local resolving set or a connected local basis of G, and this cardinality is the connected local dimension of G. For a set $W=\{w_1,w_2,...,w_k\}$ of vertices of G, the multirepresentation of vertex v of G with respect to W is the k-multiset $mr(v|W)=\{d(v,w_1),d(v,w_2),...,d(v,w_k)\}$. The set W is a multiresolving set of G if the multirepresentations of every two vertices of G with respect to W are distinct. A multiresolving set of G with minimum cardinality is a minimum multiresolving set or a multibasis of G, and this cardinality is the multidimension of G. In this work, we studied the connected local dimensions of some well-known graphs and the relationships between connected local bases and local bases in a connected graph, and some realization results. Next, the relationship between the elements in multirepresentations of vertices that belonged to the same multisimilar class was investigated. Moreover, the caterpillars were characterized with multidimension 3 and studying the multiresolving sets of symmetric caterpillars.

Keyword : Resolvability, Representations, Multirepresentations

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INTRODUCTION

In the mathematical field of graph theory, one of the problems is to provide representations of the vertices in a connected graph in such a way that distinguishing vertices have distinct representations.

1.1 Background

The *distance* from a vertex u to a vertex v in a connected graph G is the length of a shortest $u-v$ path in G, which is denoted by $d(u,v)$. For an ordered set $W=\{w_1,w_2,\ldots,w_k\}$ of vertices and a vertex v of G, the *representation* of v with respect to W is the $k-$vector $r(v\mid W)=(d(v,w_1),d(v,w_2),\ldots,d(v,w_k))$. An ordered set W is called a *resolving set* of G if every pair of two distinct vertices of G have distinct representations with respect to W. A resolving set of G containing a minimum number of vertices is called a *minimum resolving set* or a *basis* of G. The cardinality of basis of G is the *dimension* of G, which is denoted by $\dim(G)$. To illustrate this concept, consider the graph G of Figure 1.

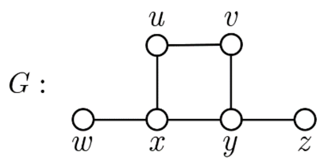

Figure 1: The graph G

For the ordered set $W_1=\{w,x\}$, since $r(u\mid W_1)=(2,1)=r(y\mid W_1)$, it follows that W_1 is not a resolving set of G. On the other hand, consider the ordered set $W_2=\{w,x,z\}$. The representations of vertices of G with respect to W_2 are

$$r(u\mid W_2)=(2,1,3),\quad r(v\mid W_2)=(3,2,2),\quad r(w\mid W_2)=(0,1,3),$$
$$r(x\mid W_2)=(1,0,2),\quad r(y\mid W_2)=(2,1,1),\quad r(z\mid W_2)=(3,2,0).$$

Since these representations are distinct, it follows that W_2 is a resolving set of G. However, W_2 is not a basis of G. To see this, consider the set $W_3 = \{w, z\}$. The representations of vertices of G with respect to W_3 are

$$\begin{aligned} &r(u \mid W_3) = (2,3), \quad r(v \mid W_3) = (3,2), \quad r(w \mid W_3) = (0,3), \\ &r(x \mid W_3) = (1,2), \quad r(y \mid W_3) = (2,1), \quad r(z \mid W_3) = (3,0). \end{aligned}$$

Thus, W_3 is a resolving set of G. Since G has no resolving set consisting of a single vertex, it follows that W_3 is a resolving set of G having minimum cardinality. Hence, W_3 is a basis of G and so $\dim(G) = 2$.

For every ordered set $W = \{w_1, w_2, \ldots, w_k\}$ of vertices of a connected graph G of order $n \geq 2$, since the only vertex of G whose representation with respect to W contains 0 in its i^{th} coordinate is w_i, it follows that the vertices of W necessarily have distinct representations. Therefore, when determining whether an ordered set W of G is a resolving set of G, we need only be concerned with the vertices of $V(G) - W$. Consequently, for a vertex v of a nontrivial connected graph G, $V(G)$ and $V(G) - \{v\}$ are resolving sets of G. This implies that the dimension of G is at most $n - 1$. Indeed, for every connected graph of order $n \geq 2$,

$$1 \leq \dim(G) \leq n - 1. \tag{1.1}$$

The concepts of resolving sets and minimum resolving sets were introduced by Slater in (1) and (2). He used a locating set for what we have called a resolving set and referred to the cardinality of a basis of a connected graph as its location number. He described the usefulness of these ideas when working with U.S. sonar and coast guard LORAN (long range aids to navigation) stations. Following Slater and others (3-5), we can think of a resolving set as the set W of vertices in a graph G so that each vertex in G is uniquely determined by its distances to the vertices of W.

To illustrate this concept, we consider a somewhat simplified example. Suppose that a certain laboratory consists of four rooms R_1, R_2, R_3 and R_4 as shown in Figure 2. The distance from R_1 to R_3 is 2 and the distance from R_2 to R_4 is also 2. The distance between all other pairs of distinct rooms is 1. The distance between a room and itself is 0. Suppose that a (red) gas sensor is placed in one of the rooms. If a gas leak occurs in one of the rooms, then the sensor is able to detect the distance from the room with the red gas sensor to the room having the gas leak. For example, suppose that the sensor is placed in R_1. If the sensor alerts us that a gas leak occurs in a room at distance 2 from R_1, then a gas leak is in R_3 since R_3 is the only one room at distance 2 from R_1. If the sensor indicates that a gas leak occurs in a room at distance 0 from R_1, then a gas leak is in R_1. However, if the sensor presents that a gas leak has occurred in a room at distance 1 from R_1, then there are two rooms R_2 and R_4 having distance 1 from R_1. For this information, we cannot tell exactly in which room a gas leak has occurred. In fact, there is no room in which the (red) gas sensor can be placed to identify the exact location of a gas leak in every instance.

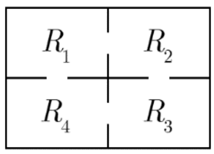

Figure 2: A laboratory consisting of four rooms

On the other hand, if we place the red and blue gas sensors in R_1 and R_2, respectively and a gas leak occurs in R_4, then the red gas sensor tells us that a gas leak occurs in a room at distance 1 from R_1, while the blue gas sensor tells us that a gas leak is in a room at distance 2 from R_2, that is, the ordered pair $(1,2)$ is produced for R_4. Since these ordered pairs are distinct for all rooms, it follows that the minimum number of gas sensors required to detect the exact location of a gas leak is 2. Care must be taken, however, as to where the two gas sensors are placed. For example, we cannot place gas sensors in R_1 and R_3 since, in this case, the ordered pairs of R_2 and R_4 are $(1,1)$. This means that we cannot distinguish the precise location of the gas leak.

The laboratory that we have just described can be modeled by a graph of Figure 3, whose vertices are the rooms and whose edges represent two rooms having distance 1.

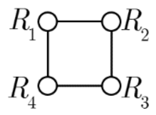

Figure 3: A graph representing a laboratory with four rooms

Harary and Melter (6) discovered these concepts independently as well but used the term *metric dimension* rather than location number, the terminology that we have adopted. These concepts were rediscovered by Johnson (7) of the Pharmacia Company while attempting to develop a capability of large datasets of chemical graphs. He and his coauthors (8) used the term resolving set for locating set and used metric dimension for location number. Wang, Miao and Liu (9) characterized the dimension of a connected graph by using metric matrix. We refer to the book (10) for graphical-theoretical notation and terminology not described in this dissertation.

1.2 Some Known Results on the Dimension of Graphs

The dimensions of some well-known classes of graphs have been determined in (1, 8, 11, 12). We state these in the next three results.

Theorem A. *Let G be a connected graph of order $n \geq 2$. Then*

(i) $\dim(G) = 1$ *if and only if* $G = P_n$, *the path of order* n,

(ii) $\dim(G) = n - 1$ *if and only if* $G = K_n$, *the complete graph of order* n,

(iii) $\dim(C_n) = 2$, *where* C_n *is the cycle of order* $n \geq 3$,

(iv) $\dim(G) = n - 2$, *where* $n \geq 4$ *if and only if* $G = K_{s,t}$, *where* $s, t \geq 1$ *or* $G = K_s + \overline{K_t}$, *where* $s \geq 1, t \geq 2$ *or* $G = K_s + (K_1 \cup K_t)$, *where* $s, t \geq 1$.

To determine the dimension of tree that is not a path, we need some additional definitions and notation. A vertex of degree at least 3 of a connected graph G is called a *major vertex* of G. Every end-vertex u of G is a *terminal vertex* of a major vertex v of G if $d(u,v) < d(u,w)$ for every other major vertex w of G. The number of terminal vertices of v is the *terminal degree* of v, which is denoted by $\mathrm{ter}(v)$. A major vertex v is called an *exterior major vertex* of G if $\mathrm{ter}(v) \geq 1$. Let $\sigma(G)$ be the sum of the terminal degrees of the major vertices of G and let $\mathrm{ex}(G)$ be the number of exterior major vertices of G. For example, consider the tree T of Figure 4. The vertices v, v_1, v_2, v_3, v_4 are five major vertices of T and the vertices $u_1, u_2, u_3, u_4, u_5, u_6, u_7, u_8$ are terminal vertices of T. Since $\mathrm{ter}(v) = 0$, $\mathrm{ter}(v_i) = 2$, where $1 \le i \le 4$, it follows that $\sigma(T) = 8$ and $\mathrm{ex}(T) = 4$.

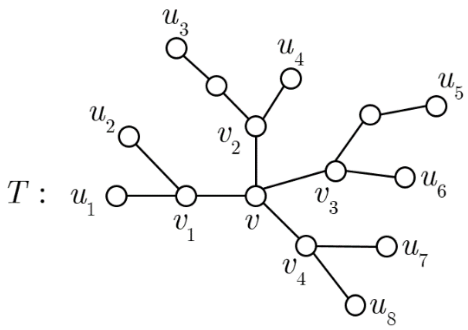

Figure 4: The tree T with $\sigma(T) = 8$ and $\mathrm{ex}(T) = 4$

Theorem B. *If T is a tree that is not a path, then* $\dim(T) = \sigma(T) - \mathrm{ex}(T)$.

Moreover, all bases of a tree that is not a path have been characterized in (12), as we state next.

Theorem C. *Let T be a tree with p exterior major vertices $v_1, v_2, \ldots, v_p$. For each integer i with $1 \le i \le p$, let $u_{i1}, u_{i2}, \ldots, u_{ik_i}$ be the terminal vertices of v_i, and let P_{ij} be the $v_i - u_{ij}$ path for $1 \le j \le k_i$. Suppose that W is a set of vertices of T. Then W is a basis of T if and only if W contains exactly one vertex from each of the path $P_{ij} - v_i$, where $1 \le j \le k_i$ and $1 \le i \le p$, with exactly one exception for each i with $1 \le i \le p$ and W contains no other vertices of T.*

1.3 Some Known Results on the Local Dimension of Graphs

Let W be an ordered set of vertices of a connected graph G. For every pair u and v of adjacent vertices in G, if $r(u \mid W) \ne r(v \mid W)$, then W is called a *local resolving set* of G. A local resolving set of G having minimum cardinality is a *minimum local resolving set* or a *local basis* of G and this cardinality is the *local dimension* of G, which is denoted by $\mathrm{ld}(G)$. To illustrate this concept, consider a connected graph G of Figure 5.

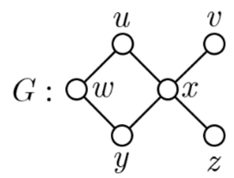

Figure 5: A connected graph G

Considering an ordered set $W_1 = \{v, w\}$, there are six representations of vertices of G with respect to W_1:

$$r(u \mid W_1) = (2,1), \quad r(v \mid W_1) = (0,3), \quad r(w \mid W_1) = (3,0),$$
$$r(x \mid W_1) = (1,2), \quad r(y \mid W_1) = (2,1), \quad r(z \mid W_1) = (2,3).$$

Observe that $r(u \mid W_1) = r(y \mid W_1)$. Then W_1 is not a resolving set of G. However, since representations of any two adjacent vertices of G with respect to W_1 are distinct, it follows that W_1 is a local resolving set of G. However, W_1 is not a local basis of G. Let $W_2 = \{u\}$. Then the representations of vertices of G with respect to W_2 are

$$r(u \mid W_2) = (0), \quad r(v \mid W_2) = (2), \quad r(w \mid W_2) = (1),$$
$$r(x \mid W_2) = (1), \quad r(y \mid W_2) = (2), \quad r(z \mid W_2) = (2).$$

For any two adjacent vertices of G, since their representations with respect to W_2 are distinct, it follows that W_2 is also a local resolving set of G. In fact, W_2 is a local basis of G and so $\mathrm{ld}(G) = 1$. Observe that W_2 is not a resolving set of G since $r(v \mid W_2) = (2) = r(y \mid W_2)$. This implies that every resolving set of G is also a local resolving set of G but every local resolving set of G need not be a resolving set of G, that is,

$$1 \le \mathrm{ld}(G) \le \dim(G) \le n - 1. \tag{1.2}$$

Okamoto, Crosse, Phinezy and Zhang (13) presented the idea of a local resolving set and the local dimension of graphs. They characterized all nontrivial connected graphs of order n having the local dimension 1, $n-2$ or $n-1$.

Theorem D. *Let G be a nontrivial connected graph of order n. Then $\mathrm{ld}(G) = n-1$ if and only if $G = K_n$ and $\mathrm{ld}(G) = 1$ if and only if G is bipartite.*

A *clique* in a graph G is a complete subgraph of G. The order of the largest clique in a graph G is its *clique number*, which is denoted by $\omega(G)$.

Theorem E. *A connected graph G of order $n \geq 3$ has local dimension $n-2$ if and only if $\omega(G) = n-1$.*

1.4 Some Known Results on the Connected Dimension of Graphs

A subgraph H of a graph G is called an *induced subgraph* of G if whenever u and v are vertices of H and uv is an edge of G, then uv is an edge of H as well. If S is a nonempty set of vertices of a graph G, then the *subgraph of G induced by S* is the induced subgraph with vertex set S. This induced subgraph is denoted by $\langle S \rangle_G$ or simply $\langle S \rangle$ if the graph G under consideration is clear. Since a connected graph G may have several resolving sets, we consider a particular resolving set W of G whose the subgraph induced by W is connected. A resolving set W of a connected graph G is called a *connected resolving set* of G if the induced subgraph $\langle W \rangle$ induced by W is connected. The minimum cardinality of a connected resolving set of G is the *connected dimension* of G, which is denoted by $\mathrm{cd}(G)$ and a connected resolving set of G having this cardinality is called a *minimum connected resolving set* or a *connected basis* of G. To illustrate this concept, consider the graph G of Figure 6.

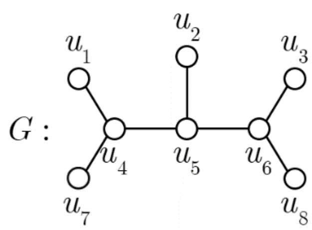

Figure 6: The graph G

Since G is a tree, it follows by Theorem C that the ordered set $W_1 = \{u_1, u_3\}$ is a basis of G. However, since $\langle W_1 \rangle$ is not connected, it follows that W_1 is not a connected resolving set of G. Let $W_2 = \{u_1, u_3, u_4, u_5, u_6\}$. Notice that $\langle W_2 \rangle = (u_1, u_4, u_5, u_6, u_3)$ is a path of order 5 and the eight representations of vertices of G with respect to W_2 are

$$\begin{aligned}
&r(u_1 \mid W_2) = (0,4,1,2,3), \quad r(u_2 \mid W_2) = (3,3,2,1,2), \quad r(u_3 \mid W_2) = (4,0,3,2,1),\\
&r(u_4 \mid W_2) = (1,3,0,1,2), \quad r(u_5 \mid W_2) = (2,2,1,0,1), \quad r(u_6 \mid W_2) = (3,1,2,1,0),\\
&r(u_7 \mid W_2) = (2,4,1,2,3), \quad r(u_8 \mid W_2) = (4,2,3,2,1).
\end{aligned}$$

Since these representations are distinct and $\langle W_2 \rangle$ is connected, it follows that W_2 is a connected resolving set of G. By Theorem C, we see that exactly one of $\{u_1, u_7\}$ and exactly one of $\{u_3, u_8\}$ must belong to every basis of G. Since there is exactly one $u_i - u_j$ path in G, where $i \in \{1,7\}$ and $j \in \{3,8\}$, it follows that every connected basis of G must contain u_4, u_5 and u_6, that is, W_2 is a connected resolving set of G having minimum cardinality. Hence, W_2 is a connected basis of G and so $\mathrm{cd}(G) = 5$.

Observe that every connected resolving set of G is a resolving set of G. On the other hand, a resolving set of a connected graph G need not be a connected resolving set of G. This implies that

$$1 \le \dim(G) \le \mathrm{cd}(G) \le n - 1. \tag{1.3}$$

The idea of connected resolving sets has appeared in (14) and used the connected resolving number $\mathrm{cr}(G)$ of G for what we have called here the connected dimension $\mathrm{cd}(G)$ of G. Some well-known graphs are characterized as we state next.

Theorem F. *Let G be a connected graph of order $n \ge 3$. Then*

(i) *if $G = P_n$, a path of order n, then $\mathrm{cd}(G) = 1$,*

(ii) *if $G = C_n$, a cycle of order n, then $\mathrm{cd}(G) = 2$,*

(iii) *$\mathrm{cd}(G) = n - 1$ if and only if $G = K_n$ or $G = K_{1,n-1}$, a complete graph or a star of order n.*

Theorem G. *For $k \ge 2$, let $G = K_{n_1, n_2, \dots, n_k}$ be a complete $k-$partite graph that is not a star. Let $n = n_1 + n_2 + \cdots + n_k$ and l be the number of one's in $\{n_i \mid 1 \le i \le k\}$. Then*

$$\mathrm{cd}(G) = \begin{cases} n - k & \text{if } l = 0, \\ n - k + l - 1 & \text{if } l \ge 1. \end{cases}$$

CHAPTER 2
THE CONNECTED LOCAL DIMENSION OF GRAPHS

We mentioned in Chapter 1 that an ordered set W of vertices of a connected graph G is a local resolving set of G if every pair of adjacent vertices of G have distinct representations with respect to W. Moreover, W is a connected resolving set of G if every pair of vertices of G have distinct representations with respect to W and the subgraph of G induced by W is connected. This idea leads us to consider a local resolving set W of G whose induced subgraph by W is connected.

2.1 Introduction

Let W be an ordered set of vertices of a connected graph G. Then W is called a *connected local resolving set* of G if W is a local resolving set of G such that the induced subgraph $\langle W \rangle$ of G is connected. A connected local resolving set of G having minimum cardinality is a *minimum connected local resolving set* or a *connected local basis* of G and this cardinality is the *connected local dimension* of G, which is denoted by $\mathrm{cld}(G)$. To illustrate this concept, consider the graph G of Figure 7.

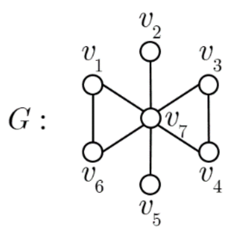

Figure 7: The graph G

We consider the representations of vertices of G with respect to the ordered set $W_1 = \{v_1, v_3\}$. Therefore, their representations with respect to W_1 are

$$\begin{aligned}
&r(v_1 \mid W_1) = (0,2), \quad r(v_2 \mid W_1) = (2,2), \quad r(v_3 \mid W_1) = (2,0),\\
&r(v_4 \mid W_1) = (2,1), \quad r(v_5 \mid W_1) = (2,2), \quad r(v_6 \mid W_1) = (1,2),\\
&r(v_7 \mid W_1) = (1,1).
\end{aligned}$$

Since any two adjacent vertices have distinct representations with respect to W_1, it follows that W_1 is a local resolving set of G. However, W_1 is not a connected local resolving set of G since $\langle W_1 \rangle$ is not connected. Then consider the ordered set $W_2 = \{v_1, v_3, v_7\}$. The representations of vertices of G with respect to W_2 are

$$\begin{aligned} &r(v_1 \mid W_2) = (0,2,1), \quad r(v_2 \mid W_2) = (2,2,1), \quad r(v_3 \mid W_2) = (2,0,1), \\ &r(v_4 \mid W_2) = (2,1,1), \quad r(v_5 \mid W_2) = (2,2,1), \quad r(v_6 \mid W_2) = (1,2,1), \\ &r(v_7 \mid W_2) = (1,1,0). \end{aligned}$$

Since representations of any two adjacent vertices of G with respect to W_2 are distinct, it follows that W_2 is a local resolving set of G. Moreover, $\langle W_2 \rangle$ is connected and so W_2 is a connected local resolving set of G. By a case-by-case analysis, it can be shown that every connected local resolving set of G must contain at least two vertices, that is, one of $\{v_1, v_6\}$ and one of $\{v_3, v_4\}$. Thus, there is no connected local resolving set of G having cardinality 2 and so W_2 is a connected local basis of G. Hence, $\mathrm{cld}(G) = 3$.

Observe that every connected local resolving set of a connected graph G is also a local resolving set of G but a local resolving set of G may or may not be a connected local resolving set of G. This implies that

$$1 \le \mathrm{ld}(G) \le \mathrm{cld}(G) \le n-1. \tag{2.1}$$

If W is a connected local resolving set of G, then $\langle W \rangle$ is connected. However, since the representations of any two vertices of G need not be distinct, it follows that W is not necessarily a connected resolving set of G. In fact, every connected resolving set of G is a connected local resolving set of G, that is,

$$1 \le \mathrm{cld}(G) \le \mathrm{cd}(G) \le n-1. \tag{2.2}$$

From (2.1) and (2.2), we obtain that

$$1 \le \mathrm{ld}(G) \le \mathrm{cld}(G) \le \mathrm{cd}(G) \le n-1. \tag{2.3}$$

For every ordered set $W = \{w_1, w_2, \ldots, w_k\}$ of vertices of a connected graph G, recall that the only vertex of G whose representation with respect to W contains 0 in its i^{th} coordinate is w_i, that is, the vertices of W necessarily have distinct representations with respect to W. On the other hand, the representations of vertices of G that do not belong to W have elements, all of which are positive. Indeed, to determine whether an ordered set W is a connected local resolving set of G, we only need to show that any two adjacent vertices in $V(G) - W$ have distinct representations with respect to W and $\langle W \rangle$ is connected.

2.2 The connected local dimensions of some well-known graphs

We determined the connected local dimensions of some well-known graphs.

Theorem 2.2.1. *Let G be a connected graph of order $n \geq 2$. Then*

(i) $\mathrm{cld}(G) = 1$ *if and only if G is a bipartite graph*,

(ii) $\mathrm{cld}(G) = n - 1$ *if and only if $G = K_n$, a complete graph of order n.*

Proof. (i) Assume that $\mathrm{cld}(G) = 1$. Then $\mathrm{ld}(G) = 1$ by (2.3). Therefore, G is bipartite by Theorem D. For converse, suppose that G is bipartite. By Theorem D, $\mathrm{ld}(G) = 1$ and so there is a 1-element local basis W of G. Indeed, W is also a connected local basis of G, that is, $\mathrm{cld}(G) = 1$.

(ii) Suppose that $\mathrm{cld}(G) = n - 1$. (2.3) implies that $\mathrm{cd}(G) = n - 1$. Thus, by Theorem F (iii), G is complete or star. If G is a star that is not complete, then G is a bipartite graph of order at least 3. By (i), $\mathrm{cld}(G) = 1$, a contradiction. Hence, G is complete. On the other hand, if $G = K_n$, then by Theorem D, $\mathrm{ld}(G) = n - 1$ and so $\mathrm{cld}(G) = n - 1$ by (2.3).

Theorem 2.2.2. *For an integer $n \geq 3$, the connected local dimension of a cycle C_n is*

$$\mathrm{cld}(C_n) = \begin{cases} 1 & \textit{if } n \textit{ is even}, \\ 2 & \textit{if } n \textit{ is odd}. \end{cases}$$

Proof. If n is even, then C_n is bipartite. By Theorem 2.2.1 (i), $\mathrm{cld}(G) = 1$. We may assume that n is odd. Let $C_n = (v_1, v_2, \ldots, v_n, v_1)$ and let $W = \{v_1, v_2\}$. Therefore, the representations of vertices in $V(C_n) - W$ are

$$r(v_i \mid W) = \begin{cases} (i-1, i-2) & \text{if } 3 \le i \le \frac{n+1}{2} \\ \left(\frac{n-1}{2}, \frac{n-1}{2}\right) & \text{if } i = \frac{n+3}{2} \\ (n-i+1, n-i+2) & \text{if } \frac{n+5}{2} \le i \le n. \end{cases}$$

Thus, W is a local resolving set of C_n. Since $\langle W \rangle$ is connected, it follows that W is a connected local resolving set of C_n and so $\mathrm{cld}(C_n) \le 2$. Since C_n is not bipartite, it follows by Theorem 2.2.1 (i) that $\mathrm{cld}(C_n) \ge 2$. Hence, $\mathrm{cld}(C_n) = 2$.

Observe that if G' is a graph obtained by adding a pendant edge to a connected graph $G,$ then it is easy to verify that $\mathrm{cld}(G') = \mathrm{cld}(G)$. However, if a vertex v is added to a connected graph G such that more than one edge is incident with v, then the connected local dimension of the resulting graph can stay the same, decrease, or increase significantly. For example, for $n \ge 3$, $1 \le \mathrm{cld}(C_n) \le 2$. Consider the connected local dimension of a *wheel* $W_n = C_n + K_1,$ where $n \ge 3$. Clearly, $\mathrm{cld}(W_3) = 3,$ $\mathrm{cld}(W_4) = \mathrm{cld}(W_5) = 2$ and $\mathrm{cld}(W_6) = 3$. However, for $n \ge 7$, the connected local dimension of a wheel W_n increase with n as we show next.

In $W_n = C_n + K_1$, let $C_n = (v_1, v_2, \ldots, v_n, v_1)$, where $n \ge 7$, and let v be the central vertex of W_n. Let S be a set of two or more vertices of C_n, let v_i and v_j be two distinct vertices of S, and let P and P' denote the two distinct $v_i - v_j$ paths determined by C_n. If either P or P', say P, contains only two vertices of S (namely, v_i and v_j), then we refer to v_i and v_j as *neighboring vertices* of S and the set of vertices of P that belong to $C_n - \{v_i, v_j\}$ as the *gap* of S (determined by v_i and v_j). The two gaps of S determined by a vertex of S and its two neighboring vertices will be referred to as *neighboring gaps*. Consequently, if $|S| = r$, then S has r gaps, some of which may be empty.

Observe that every connected local basis of W_n does not contain v since $d(v, v_i) = 1$ for all integer i with $1 \le i \le n$. The next theorem presents a necessary and sufficient condition for a set W to be a local resolving set of W_n.

Theorem 2.2.3. *Let W be a set of vertices of a wheel $W_n = C_n + K_1$, where $n \ge 7$. Then W is a local resolving set of W_n if and only if every gap of W contains at most three vertices of C_n.*

Proof. Assume, to the contrary, that there is a gap of W containing at least four vertices of C_n. Then there are two adjacent vertices u and u' in this gap such that $d(u,w) = d(u',w) = 2$ for all $w \in W - \{v\}$. Therefore, $r(u \mid W) = r(u' \mid W)$, which is impossible. To show the converse, suppose that every gap of W contains at most three vertices of C_n. Since $n \geq 7$, it follows that W contains at least three vertices of C_n. Since v is adjacent to every vertex of C_n, it follows that the representation of v and any vertices of C_n with respect to W are distinct. Therefore, we need to consider only two adjacent vertices in each gap of W. Let u and w be two adjacent vertices of C_n such that $u, w \notin W$. Thus, u and w belong to a gap of size 2 or 3. If u and w belong to a gap of size 2, then for $1 \leq i \leq n$, we may assume that $v_i, u = v_{i+1}, w = v_{i+2}, v_{i+3}$ are consecutive vertices of C_n, where $v_i, v_{i+3} \in W$ and addition is performed modulo n. Since $d(v_{i+1}, v_i) = 1$ and $d(v_{i+2}, v_i) = 2$, it follows that the representations of v_{i+1} and v_{i+2} with respect to W are distinct. If u and w belong to a gap of size 3, then we may assume that $v_i, v_{i+1}, v_{i+2}, v_{i+3}, v_{i+4}$ are vertices of C_n, where $v_i, v_{i+4} \in W$ and $v_{i+1}, v_{i+2}, v_{i+3} \notin W$. Without, loss of generality, let $u = v_{i+1}$ and $w = v_{i+2}$. Since $d(v_{i+1}, v_i) = 1$ and $d(v_{i+2}, v_i) = 2$. Thus, $r(u \mid W) \neq r(w \mid W)$. Hence, W is a local resolving set of W_n.

Recall that for $n \geq 7$, every local basis of W_n contains no central vertex. However, every connected local basis of W_n must contain the central vertex. It is shown in the next result.

Lemma 2.2.4. *Every connected local basis of a wheel W_n, where $n \geq 7$ must contain the central vertex*.

Proof. Assume, to the contrary, that there is a connected local basis W of W_n not containing the central vertex v. Then W consists of consecutive vertices in C_n. Without, loss of generality, let $W = \{v_1, v_2, \ldots, v_k\}$. By Theorem 2.2.3, it implies that $k \geq n - 3$. By the argument similar to the one used for the proof of Theorem 2.2.3, the set $W' = \{v, v_1, v_4, v_5, \ldots, v_k\}$ is a local resolving set of W_n having cardinality $k-1$, contradicting the assumption that W is a connected local basis of W_n.

We are now prepared to present the connected local dimension of a wheel W_n, where $n \ge 7$.

Theorem 2.2.5. *Let W_n be a wheel, where $n \ge 7$. Then* $\mathrm{cld}(W_n) = \left\lceil \frac{n}{4} \right\rceil + 1$.

Proof. By Theorem 2.2.3 and Lemma 2.2.4, we obtain that $\mathrm{cld}(W_n) \ge \left\lceil \frac{n}{4} \right\rceil + 1$. It remains to verify that $\mathrm{cld}(W_n) \le \left\lceil \frac{n}{4} \right\rceil + 1$. Let $W = \{v_i \in V(C_n) \mid i \equiv 1 (\mathrm{mod}\, 4)\} \cup \{v\}$ with $|W| = \left\lceil \frac{n}{4} \right\rceil + 1$. Since every gap of W contains at most three vertices from C_n, it follows by Theorem 2.2.3 that W is a local resolving set of W_n. Moreover, since W contains the central vertex v, it follows that $\langle W \rangle$ is connected and so W is a connected local resolving set of W_n. Therefore, $\mathrm{cld}(W_n) \le \left\lceil \frac{n}{4} \right\rceil + 1$. Hence $\mathrm{cld}(W_n) = \left\lceil \frac{n}{4} \right\rceil + 1$.

2.3 Graphs with prescribed connected local dimensions and other parameters

The *open neighborhood* or the *neighborhood* of a vertex u of a connected graph G is the set of all vertices that are adjacent to u, which is denoted by $N(u) = \{v \in V(G) \mid uv \in E(G)\}$. The *closed neighborhood* $N[u]$ of u is defined as $N(u) \cup \{u\}$. Two vertices u and v of G are *twins* if $N(u) - \{v\} = N(v) - \{u\}$. If $N[u] = N[v]$, then u and v are called *true twins* while if $N(u) = N(v)$, then u and v are called *false twins*. We define a relation on $V(G)$ by u is related to v if they are true twins. This relation is an equivalence relation and, as such, this relation partitions $V(G)$ into equivalence classes which are called *true twin equivalence classes* or simply *true twin classes* on $V(G)$. Observe that if G contains l distinct true twin classes $U_1, U_2, \ldots, U_l$, then every connected local resolving set of G must contain at least $|U_i| - 1$ vertices from U_i for each integer i with $1 \le i \le l$. This observation has been described in (13) as we state next.

Proposition H. *Let G be a connected graph having l true twin classes $U_1, U_2, \ldots, U_l$. Then every local resolving set of G must contain $|U_i| - 1$ vertices from each U_i, where $1 \le i \le l$. Moreover,* $\mathrm{ld}(G) \ge \sum_{i=1}^{l} |U_i| - l$.

We have seen that if G is a connected graph of order n with $\mathrm{ld}(G) = a$ and $\mathrm{cld}(G) = b$, then $1 \le a \le b \le n-1$ by (2.1). A common problem concerns whether every three integers a, b and n with $1 \le a \le b \le n-1$ are realizable as the local dimension, connected local dimension and order of some graph as we show next.

Theorem 2.3.1. *Let a, b and n be integers with $n \ge 4$. Then there exists a connected graph G of order n with $\mathrm{ld}(G) = a$ and $\mathrm{cld}(G) = b$ if and only if a, b, n satisfy one of the following*:

(i) $a = b = n-1$,

(ii) $a = b = 1$, *and*

(iii) $2 \le a \le b \le n-2$.

Proof. Assume that there exists a connected graph G of order n with $\mathrm{ld}(G) = a$ and $\mathrm{cld}(G) = b$. By (2.1), we obtain that $1 \le a \le b \le n-1$. If $b = n-1$, then G is a complete graph K_n. Thus, $a = b = n-1$. If $a = 1$, then G is a bipartite graph. Therefore, $a = b = 1$. For otherwise, $2 \le a \le b \le n-2$. Hence, if G is a connected graph of order n with $\mathrm{ld}(G) = a$ and $\mathrm{cld}(G) = b$, then a, b and n must satisfy one of (i), (ii) and (iii). It remains to verify the converse. If $a = b = n-1$, then let G be a complete graph K_n and the result is true. If $a = b = 1$, then let G be a path P_n. Thus, the graph G has the desired properties. We may assume that $2 \le a \le b \le n-2$. We consider two cases.

Case 1. $a = b$.

Let G' be a graph obtained from a complete graph K_a with vertex set $\{u_1, u_2, \ldots, u_a\}$ and a path $P_{n-a} = (v_1, v_2, \ldots, v_{n-a})$ by joining v_1 to every vertex of K_a. Since $V(K_a)$ is a true twin class of G', it follows by Proposition H that every local resolving set of G' must contain at least $a-1$ vertices from $V(K_a)$. However, if a set W contains only $a-1$ vertices from $V(K_a)$, then W does not contain u_i for some integer i with $1 \le i \le a$ and so $r(u_i \mid W) = r(v_1 \mid W) = (1,1,\ldots,1)$. Therefore, G'

contains no local resolving set of cardinality $a-1$, that is, $\mathrm{ld}(G') \ge a$. Since the representation of each vertex of P_{n-a} is $r(v_j \mid V(K_a)) = (j, j, \ldots, j)$, where $1 \le j \le n-a$, it follows that $V(K_a)$ is a local resolving set of G' having cardinality a, that is, $V(K_a)$ is a local basis of G'. Moreover, $V(K_a)$ is also a connected local basis of G'. Hence, $\mathrm{ld}(G') = \mathrm{cld}(G') = a$.

Case 2. $a < b$.

Let G be a graph obtained from a complete graph K_a with vertex set $\{u_1, u_2, \ldots, u_a\}$ and two path $P_{b-a+1} = (v_1, v_2, \ldots, v_{b-a+1})$ and $P_{n-b-1} = (w_1, w_2, \ldots, w_{n-b-1})$ by joining v_1 to every vertex of K_a, and w_1 to both v_{b-a} and v_{b-a+1}. Since $V(K_a)$ is a true twin class of G, it follows by Proposition H that every local resolving set of G must contain at least $a-1$ vertices from $V(K_a)$. However, every set consisting of $a-1$ vertices from $V(K_a)$ is not a local resolving set of G since the representations of v_{b-a+1} and w_1 with respect to this set are the same. Thus, every local resolving set of G contains at least a vertices. It is routine to verify that every local resolving set of G must contain at least one vertex from $\{v_{b-a+1}\} \cup V(P_{n-b-1})$. Then the set $\left(V(K_a) - \{u_1\}\right) \cup \{v_{b-a+1}\}$ is a minimum local resolving set of G. Hence, $\mathrm{ld}(G) = a$. Since every connected local resolving set of G is also a local resolving set of G, it follows that every connected local resolving set of G must contain at least $a-1$ vertices from $V(K_a)$ and at least one vertex from $\{v_{b-a+1}\} \cup V(P_{n-b-1})$. Therefore, every connected local resolving set of G contains $v_1, v_2, \ldots, v_{b-a}$. In fact, the set $\left(V(K_a) - \{u_1\}\right) \cup V(P_{b-a+1})$ is a connected local basis of G, that is, $\mathrm{cld}(G) = b$.

We know by (2.2) that if G is a connected graph of order n with $\mathrm{cld}(G) = b$ and $\mathrm{cd}(G) = c$, then $1 \le b \le c \le n-1$. Next, we show that for any integers b, c and n with $1 \le b \le c \le n-1$ are realizable as the connected local dimension, connected dimension and order of some graph.

Theorem 2.3.2. *Let b, c and n be integers with $n \ge 4$. Then there exists a connected graph G of order n with $\mathrm{cld}(G) = b$ and $\mathrm{cd}(G) = c$ if and only if b, c, n satisfy one of the following*:

(i) $b = c = n-1$,

(ii) $b = 1$ *and* $1 \leq c \leq n-1$, *and*

(iii) $2 \leq b \leq c \leq n-2$.

Proof. Assume that there exists a connected graph of order n with $\mathrm{cld}(G) = b$ and $\mathrm{cd}(G) = c$. By (1.2), we obtain that $1 \leq b \leq c \leq n-1$. If $b = n-1$, then $c = n-1$ by (1.2). If $b = 1$, then $1 \leq c \leq n-1$ by (1.2). If $2 \leq b \leq n-2$, then G is neither a star nor a complete graph and so $2 \leq b \leq c \leq n-2$. Hence, if G is a connected graph of order n with $\mathrm{cld}(G) = b$ and $\mathrm{cd}(G) = c$, then b, c and n must satisfy one of (i), (ii) and (iii). It remains to verify the converse. If $b = c = n-1$, then let G be a complete graph K_n and the result is true. Next, assume that $b = 1$ and $1 \leq c \leq n-1$. For $c = 1$, let G be a path P_n; while for $c = n-1$, let G be a star $K_{1,n-1}$. Since $\mathrm{cld}(P_n) = \mathrm{cd}(P_n) = 1$, and $\mathrm{cld}(K_{1,n-1}) = 1$ and $\mathrm{cd}(K_{1,n-1}) = n-1$, it follows that the result holds for $b = 1$ and $c = 1, n-1$. For $2 \leq c \leq n-2$, let G be a graph obtained from a complete bipartite graph $K_{2,c-1}$ with partite set $U = \{u_1, u_2\}$ and $U' = \{w_1, w_2, \ldots, w_{c-1}\}$, and a path $P_{n-c-1} = (v_1, v_2, \ldots, v_{n-c-1})$ by joining v_1 to both u_1 and u_2. Since G is bipartite, it follows that $\mathrm{cld}(G) = 1$. It is routine to show that the set $V(K_{2,c-1}) - \{u_2\}$ is a connected basis of G. Therefore, $\mathrm{cd}(G) = c$. Hence, the result holds for $b = 1$ and $2 \leq c \leq n-2$. Now assume that $2 \leq b \leq c \leq n-2$. We consider two cases.

Case 1. $b = c$.

The graph G' of the proof for Theorem 2.3.1 has $\mathrm{cld}(G') = b$ with a connected local basis $V(K_b)$. In fact, $V(K_b)$ is also a connected basis of G', that is, $\mathrm{cd}(G') = b$.

Case 2. $b < c$.

Let G be a graph obtained from a complete graph K_b with vertex set $\{u_1, u_2, \ldots, u_b\}$, a star $K_{1,c-b}$ with vertex set $\{v, v_1, v_2, \ldots, v_{c-b}\}$ and a path $P_{n-c-1} = (w_1, w_2, \ldots, w_{n-c-1})$ by joining the central vertex v of $K_{1,c-b}$ to w_1 and every vertex of K_b. It is immediate that the set $V(K_b)$ is a connected local basis of G. Therefore, $\mathrm{cld}(G) = b$. Moreover, the set $\left(V(K_b) - \{u_1\}\right) \cup V(K_{1,c-b})$ is a connected basis of G, that is, $\mathrm{cd}(G) = c$.

2.4 Connected local bases and local bases in graphs

In this section, we study the relationship between connected local bases and local bases in a connected graph G. Certainly, if W is a local resolving set of $G,$ then a set W' containing W is also a local resolving set of $G.$ Therefore, if W is a local basis of G such that $\langle W \rangle$ is disconnected, then surely there is a smallest superset W' of W for which $\langle W' \rangle$ is connected. This suggests the following question: Does there exist a graph with a connected local basis not containing any local bases? The answer to this question is given in the next result.

Theorem 2.4.1. There is an infinite class of connected graphs G such that some connected local bases of G contain a local basis of G and others contain no local basis of G.

Proof. Let G be a graph obtained from a complete graph K_a of order $a \geq 2$ with vertex set $\{u_1, u_2, \ldots, u_a\}$, a cycle $C_4 = (v_1, v_2, v_3, v_4, v_1)$ and a path $P_3 = (w_1, w_2, w_3)$ by joining v_1 to every vertex of K_a and joining w_1 and w_3 to v_1, v_4 and v_2, v_3, respectively. A graph G is shown Figure 8.

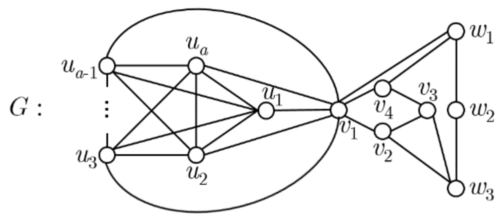

Figure 8: A graph G

We first verify that the set $B = \{u_1, u_2, .., u_{a-1}\} \cup \{w_2\}$ is a local basis of $G.$ We can show, by a case-by-case analysis, that B is a local resolving set of $G.$ Next, we claim that B is a local resolving set of minimum cardinality. Assume, to the contrary, that there is a local resolving set W of G having cardinality at most $a-1$. Since $V(K_a)$ is a true twin class of $G,$ it follows that every local resolving set of G must contain at least $a-1$ vertices of K_a. Therefore, W consists of $a-1$ vertices of K_a. However, v_4 and w_1 are adjacent and $d(v_4, u_i) = d(w_1, u_i)$ for each integer i with $1 \leq i \leq a$. This is a contradiction. Hence, B is a local basis of G and so $\mathrm{ld}(G) = a$. Second, we determine that $\text{cld}(G) = a + 2$. In order to do this, we claim that $\text{cld}(G) \geq a + 2$. Suppose, contrary to our claim, that there is a connected local resolving set W' of G having cardinality $a + 1$. Recall that every connected local basis of G must contain at least $a - 1$ vertices of K_a. We consider two cases.

Case 1. $V(K_a) \subseteq W'$.

Since $\langle W' \rangle$ is connected and $|W'| = a + 1,$ it follows that $W' = V(K_a) \cup \{v_1\}$. However, since v_4 is adjacent to w_1 and $r(v_4 \mid W') = r(w_1 \mid W'),$ it follows that W' is not a connected resolving set of $G,$ which is a contradiction.

Case 2. $V(K_a) \not\subset W'$.

Since $\langle W' \rangle$ is connected and $|W'| = a + 1$, it follows that W' contains v_1 and one vertex from $\{v_2, v_4, w_1\}$. If W' contains v_2 or w_1, then $r(v_3 \mid W') = r(w_3 \mid W')$. If W' contains v_4, then $r(w_2 \mid W') = r(w_3 \mid W')$. Therefore, W' is not a connected local resolving set of $G.$ This is also a contradiction.

Thus, $\text{cld}(G) \geq a + 2$. On the other hand, the sets $S_1 = \{u_1, u_2, \ldots, u_{a-1}\} \cup \{v_1, w_1, w_2\}$ and $S_2 = \{u_1, u_2, \ldots, u_{a-1}\} \cup \{v_1, v_4, w_1\}$ are connected local resolving sets of $G.$ Therefore, $\text{cld}(G) \leq a + 2$. Hence, $\text{cld}(G) = a + 2$.

Last, it can be verified that every local basis of G contains exactly $a - 1$ vertices of K_a and exactly one vertex from $\{v_3, w_2\}$. Observe that the connected local basis S_1 contains the local basis B of $G,$ while the connected local basis S_2 contains no local basis of $G.$

From the previous theorem, there is a connected graph having many connected local bases. This leads us to determine a connected graph G having a unique connected local basis. It has been shown in (13) that there is a connected graph with a unique local basis. In fact, there is a connected graph with a unique connected local basis as we show next.

Theorem 2.4.2. *For $k \geq 3$, there exists a graph with a unique connected local basis of cardinality $k + 1$.*

Proof. Let G_1 be a complete graph K_{2^k} with vertex set $U = \{u_0, u_1, \ldots, u_{2^k - 1}\}$, and let G_2 be an empty graph $\overline{K}_k$ with vertex set $W = \{w_{k-1}, w_{k-2}, \ldots, w_0\}$. Then the graph G

is obtained from G_1 and G_2 by adding edges between U and W as follows. Let each integer j for $1 \le j \le 2^k - 1$ be expressed in its base 2 (binary) representation. Thus, each such j can be expresses as a sequence of k coordinates, that is, a k-vector, where the rightmost coordinate represents the value (either 0 or 1) in the 2^0 position, the coordinate to its immediate left is the value in the 2^1 position, etc. For integers i and j with $0 \le i \le k-1$ and $0 \le j \le 2^k - 1$, we join w_i and u_j if and only if the value in the 2^i position in the binary representation of j is 1. For example, Figure 9 shows the edges joining between U and W in the graph G for $k = 3$.

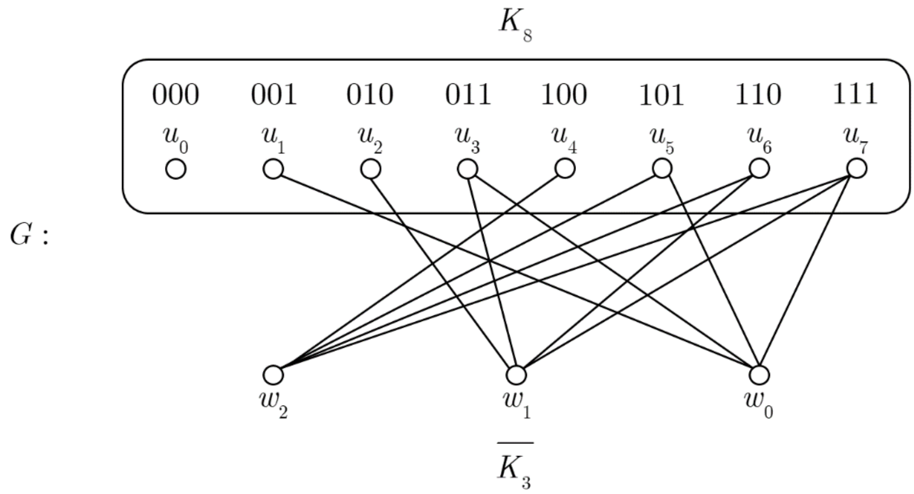

Figure 9: A graph G for $k = 3$

It was shown in (13) that W is a unique local basis of G. Therefore, there is no connected local basis of G having cardinality k, that is, $\mathrm{cld}(G) \ge k+1$. Since W is a local basis of G, it follows that $W' = W \cup \{u_{2^{k-1}}\}$ is a connected local resolving set of G. In fact, W' is a connected local basis of G.

It remains only to show that G has no other connected local basis. If $U' \subseteq U$ and $|U'| = k+1$, then $|U - U'| = 2^k - k - 1 \ge 2$. Since the distance of every two vertices of U is 1, it follows that there are at least two adjacent vertices of $U - U'$ having the same representation with respect to U' and so U' is not a connected local resolving set of G, Thus, every connected local resolving set of G must contain at least one vertex of W. Suppose that $B \ne W'$ is a connected local basis of G. Therefore,

$B = U'' \cup W''$, where $U'' \subseteq U$ and $W'' \subseteq W$. If $|W''| = k$, then B does not contain u_{2^k-1}. Therefore, $\langle B \rangle$ is not connected, which is impossible. If $|W''| \leq k-1$, then U'' contains at least two vertices. We may therefore assume that $|U''| = i \geq 2$. Then $|W''| = k-i+1$. Since every vertex of $U - U''$ has distance 1 from every vertex of U'', it follows that there are at most 2^{k-i+1} distinct representations of vertices of $U - U''$ with respect to B. However, since $2^k - i > 2^{k-i+1}$, there are two vertices of $U - U''$ such that their representations with respect to B are the same, contradicting the fact that B is a connected local basis of G. Hence, W' is a unique connected local basis of G.

CHAPTER 3
THE MULTIDIMENSION OF GRAPHS

As described in (8), all connected graphs G contain an ordered set W of vertices of G such that each vertex of G is distinguished by a $k-$vector, known as a representation, consisting of its distance from the vertices in $W.$ It may also occur that some graph contains a set W' with property that the vertices of graph have uniquely distinct $k-$multisets containing their distances from each of the vertices of $W'.$ In this section, we study the existence of such a set of connected graphs.

3.1 Introduction

A *multiset* is a generalization of the concept of a set, which is like a set except that its members need not to be distinct. For example, the set $\{a,b,a\}$ is the same as the set $\{a,b\}$ but not so for the multiset. The multiset $M=\{a,a,1,2,1,b,a,2\}$ has 8 elements of 4 different types: 3 of type a, 2 of type 1, 2 of type 2 and 1 of type b. Then the multiset is usually indicated by specifying the number of times different types of elements occur in it. Therefore, the multiset M can be written by $M=\{3\cdot a,2\cdot 1,2\cdot 2,1\cdot b\}$. The numbers $3,2,2$ and 1 are called the *repetition numbers* of the multiset M. In particular, a set is a multiset having all repetition numbers equal to 1.

Let $W=\{w_1,w_2,\dots,w_k\}$ be a set of vertices of a connected graph $G.$ The *multirepresentation* of a vertex u of G with respect to W is the $k-$multiset

$$mr(u\mid W)=\{d(u,w_1),d(u,w_2),\dots,d(u,w_k)\}.$$

The set W is called a *multiresolving set* of G if every two distinct vertices of G have distinct multirepresentations with respect to $W.$ A multiresolving set of G of minimum cardinality is a *minimum multiresolving set* or a *multibasis* of G and this cardinality is the *multidimension* of $G,$ which is denoted by $\dim_M(G)$. For example, consider the cycle C_6 of Figure 10.

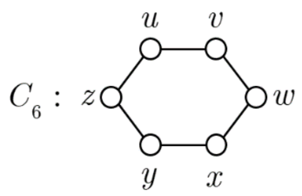

Figure 10: The cycle C_6

As we know, the set $W_1 = \{u, v\}$ is a basis of C_6. However, the set W_1 is not a multiresolving set of C_6 since $mr(u \mid W_1) = \{0,1\} = mr(v \mid W_1)$. Then we consider the set $W_2 = \{u, v, x\}$. The six multirepresentations of vertices of C_6 are

$$mr(u \mid W_2) = \{0,1,3\}, \quad mr(v \mid W_2) = \{0,1,2\}, \quad mr(w \mid W_2) = \{1,1,2\},$$
$$mr(x \mid W_2) = \{0,2,3\}, \quad mr(y \mid W_2) = \{1,2,3\}, \quad mr(z \mid W_2) = \{1,2,2\}.$$

Since these six multirepresentations are distinct, it follows that W_2 is a multiresolving set of C_6. In fact W_2 is also a multibasis of C_6 and so $\dim_M(C_6) = 3$.

Not all connected graphs have a multiresolving set and so $\dim_M(G)$ is not defined for all connected graphs G. For instant, a star $K_{1,s}$ $(s \geq 3)$ contains no multiresolving set. To see this, suppose that W is a multiresolving set of $K_{1,s}$. Then there are two end-vertices u and v of $K_{1,s}$ such that both u and v belong to either W or $V(K_{1,s}) - W$. However, there is a contradiction in both cases since $d(u,w) = d(v,w)$ for all $w \in V(K_{1,s}) - \{u, v\}$, implying that $K_{1,s}$ contains no multiresolving set. On the other hand, if a connected graph G contains a multiresolving set, then this multiresolving set is also a resolving set of G. This implies that

$$1 \leq \dim(G) \leq \dim_M(G) \leq n \tag{3.1}$$

For every set W of vertices of a connected graph G, the vertices of G whose multirepresentations with respect to W contain 0, are vertices in W. On the other hand, the multirepresentations of vertices of G which do not belong to W have elements, all of which are positive. In fact, to determine whether a set W is a multiresolving set of G, the vertex set $V(G)$ can be partitioned into W and $V(G) - W$

to examine whether the vertices in each subset have distinct multirepresentations with respect to W.

The idea of the multidimension of a connected graph was introduced by Saenpholphat (15) who showed that there is no connected graph with multidimension 2. Moreover, the multidimensions of some well-known graphs have been determined. Simanjuntak, Vetrik and Mulia (16) discovered this concept independently and used a notation $md(G)$ for a multidimension of a connected graph G.

3.2 Preliminaries

Recall that two vertices u and v of a connected graph G are twins if $N(u)-\{v\} = N(v)-\{u\}$. Actually, u and v are twins if and only if $d(u,x) = d(v,x)$ for all $x \in V(G)-\{u,v\}$. Therefore, they are said to be *distance-similar*. Certainly, distance similarity in G is an equivalence relation on $V(G)$ producing a partition of the vertex set of G into equivalence classes, called *distance-similar equivalence classes*, or simply *distance-similar classes*. For example, consider a complete bipartite graph $K_{r,s}$ $(r \geq 1, s \geq 2)$ with partite sets U and V. Every pair of vertices in the same partite set are distance-similar. Then the distance-similar classes in $K_{r,s}$ are its partite sets U and V. The following results were obtained in (15) showing the usefulness of the distance-similar classes to determine the multidimension of a connected graph.

Theorem I. *Let G be a connected graph such that $\dim_M(G)$ is defined. If U is a distance-similar class on $V(G)$ with $|U| = 2$, then every multiresolving set of G contains exactly one vertex of U.*

Theorem J. *If U is a distance-similar class on the set of vertices $V(G)$ in a connected graph G with $|U| \geq 3$, then $\dim_M(G)$ is not defined.*

The next two theorems were presented in (15, 16) that a path is the only one of connected graphs with multidimension 1 and every multiresolving set of a connected graph cannot contain only two vertices.

Theorem K. *Let G be a connected graph of order n. Then $\dim_M(G) = 1$ if and only if $G = P_n$, a path of order n.*

Theorem L. *A connected graph has no multiresolving set of cardinality* 2.

For a connected graph G, if W is a multiresolving set of $G,$ then all vertices of G have distinct multirepresentations with respect to $W.$ This leads us to the fact that W is also a multiresolving set of $G-v,$ where v is an end-vertex of G that is not in $W.$ We present this idea as follows.

Theorem 3.2.1. *Let G be a connected graph such that $\dim_M(G)$ is defined, and let W be a multiresolving set of $G.$ If v is an end-vertex of G such that $v \notin W,$ then W is a multiresolving set of $G-v.$*

Proof. Assume that v is an end-vertex of $G.$ Let $W=\{w_1,w_2,\ldots,w_k\}$ be a multiresolving set of G that does not contain v. Then

$$mr_G(x \mid W)=\{d_G(x,w_1),d_G(x,w_2),\ldots,d_G(x,w_k)\}$$

and

$$mr_G(y \mid W)=\{d_G(y,w_1),d_G(y,w_2),\ldots,d_G(y,w_k)\}$$

are not the same for all vertices x and y of $G.$ Since v does not belong to $W,$ it follows that

$$mr_{G-v}(x \mid W)=\{d_{G-v}(x,w_1),d_{G-v}(x,w_2),\ldots,d_{G-v}(x,w_k)\}=mr_G(x \mid W)$$

and

$$mr_{G-v}(y \mid W)=\{d_{G-v}(y,w_1),d_{G-v}(y,w_2),\ldots,d_{G-v}(y,w_k)\}=mr_G(y \mid W),$$

that is, $mr_{G-v}(x \mid W)\neq mr_{G-v}(y \mid W)$ for all vertices x and y of $G-v$. Hence, W is a multiresolving set of $G-v$.

The following is an immediate corollary of Theorem 3.2.1.

Corollary 3.2.2. *Let G be a connected graph such that $\dim_M(G)$ is defined, and let W be a multiresolving set of G. If $v_1,v_2,\ldots,v_t$ are end-vertices of G such that $v_1,v_2,\ldots,v_t \notin W,$ then W is a multiresolving set of $G-\{v_1,v_2,\ldots,v_t\}$.*

Proof. Assume that $v_1,v_2,\ldots,v_t$ are end-vertices of G. Let W is a multiresolving set of G that does not contain $v_1,v_2,\ldots,v_t$. Theorem 3.2.1 implies that W is a multiresolving set of $G-v_1$. By the same reasoning, W is a multiresolving set of $(G-v_1)-v_2$ and so W is a multiresolving set of $G-\{v_1,v_2,\ldots,v_t\}$.

Next, we present a useful necessary condition for a set to be a multiresolving set of a tree.

Proposition 3.2.3. *Let T be a tree of order at least 3 containing a vertex u. If W is a multiresolving set of T, then W contains at least one vertex from each of $\deg_T u$ components of $T-u$, with one possible exception.*

Proof. We see that $T-u$ has only one component if and only if u is an end-vertex of T. Then we may assume, to the contrary, that there is a vertex u of degree at least 2 such that $T-u$ has two components X and Y containing no vertex of W. Then there are two vertices x of X and y of Y that are adjacent to u in T. Thus, $d(x,w)=d(u,w)+1=d(y,w)$ for all vertices w of W. This implies that $mr(x \mid W)=mr(y \mid W)$ and so W is not a multi resolving set of T, a contradiction.

We are able to determine all pairs k,n of integers with $k \geq 3$ and $n \geq 3(k-1)$ which are realizable as the multidimension and the order of some connected graph. In order to do this, we present an additional notation. For integers a and b, let $[a,b]$ be a multiset such that

$$[a,b]=\begin{cases}\{a,a+1,\ldots,b-1,b\} & \text{if } a<b,\\ \{a\} & \text{if } a=b,\\ \varnothing & \text{if } a>b.\end{cases}$$

Such a multiset is referred to as a *consecutive multiset* of integers a and b.

Theorem 3.2.4. *For every pair k,n of integers with $k \geq 3$ and $n \geq 3(k-1)$, there is a connected graph G of order n with $\dim_M(G)=k$.*

Proof. Let k and n be integers with $k \geq 3$ and $n \geq 3(k-1)$. We consider two cases.

Case 1. $n=3(k-1)$.

Let G be a graph obtained from the path $P_{k-1}=(u_1,u_2,\ldots,u_{k-1})$ by adding the $2(k-1)$ new vertices v_i and w_i for $1 \leq i \leq k-1$ and joining v_i and w_i to u_i, as it is shown in Figure 11. Then the order of G is $n=3(k-1)$.

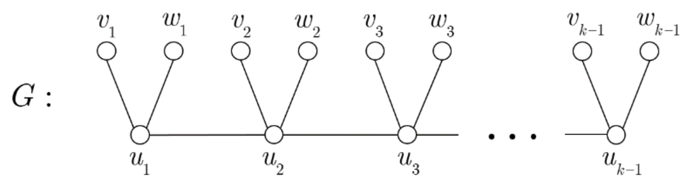

Figure 11: A connected graph G in Case 1

First, we claim that there is no multiresolving set of G having cardinality at most $k-1$. Assume, to the contrary, that there is a multiresolving set S of G such that $|S| \leq k-1$. Since a set $V_i = \{v_i, w_i\}$ for $1 \leq i \leq k-1$ is a distance-similar equivalence class of G, it follows by Theorem I that S contains exactly one vertex of V_i. We may assume, without loss of generality, that $w_i \in S$ for $1 \leq i \leq k-1$. Thus, $|S| = k-1$. Since $d(w_1, w_i) = d(w_{k-1}, w_{k-i})$ for all $1 \leq i \leq k-1$, it follows that $mr(w_1 \mid S) = mr(w_{k-1} \mid S)$ and so a set $S = \{w_1, w_2, \ldots, w_{k-1}\}$ is not a multiresolving set of G, thereby producing a contradiction. Hence, $\dim_M(G) \geq k$. Next, we claim that a set $W = \{w_1, w_2, \ldots, w_{k-1}\} \cup \{u_1\}$ is a multiresolving set of G. For a vertex $x \in W$, the multirepresentations of x with respect to W is

$$mr(x \mid W) = \begin{cases} \{0, i\} \cup [3, i+1] \cup [3, k-i+1] & \text{if } x = w_i \ (1 \leq i \leq k-1) \\ [0, k-1] & \text{if } x = u_1. \end{cases}$$

For $2 \leq i \leq k-1$, the multirepresentations of u_i with respect to W is

$$mr(u_i \mid W) = \{1, i-1\} \cup [2, i] \cup [2, k-i].$$

For $1 \leq i \leq k-1$, the multirepresentations of v_i with respect to W is

$$mr(v_i \mid W) = \{2, i\} \cup [3, i+1] \cup [3, k-i+1].$$

Therefore, W is a multiresolving set of G with $|W| = k$. Hence, $\dim_M(G) = k$.

Case 2. $n > 3(k-1)$.

Let H be a graph obtained from the graph G in Case 1 by adding the path $P = (x_1, x_2, \ldots, x_{n-3(k-1)})$ and joining x_1 to v_{k-1} and w_{k-1}, as it is shown in Figure 12.

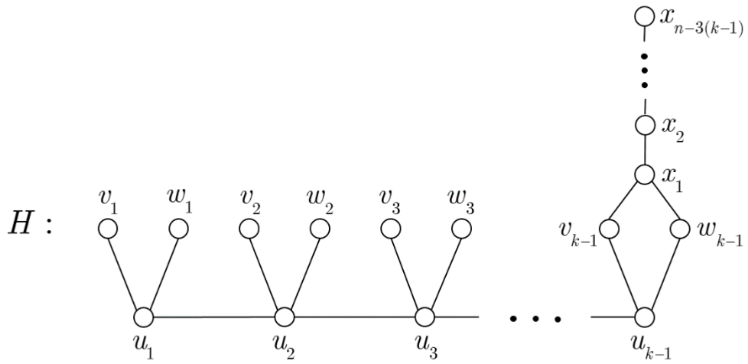

Figure 12: A connected graph H in Case 2

By a similar argument to the one used in Case 1, it is shown that there is no l-multiresolving set of H with $1 \le l \le k-1$. We claim that a set $W = \{w_1, w_2, \ldots, w_{k-1}\} \cup \{u_1\}$ is a multiresolving set of H. For vertices in $V(H) - \{x_1, x_2, \ldots, x_{n-3(k-1)}\}$, their multirepresentations with respect to W are the same as in Case 1. For $1 \le i \le n-3(k-1)$, the multirepresentations of x_i with respect to W is

$$mr(x_i \mid W) = \{i, i+k-1\} \cup [i+3, i+k].$$

Hence, W is a multiresolving set of H with $|W| = k$ and so $\dim_M(H) = k$.

3.3 The multisimilar classes of graphs

In this section, we investigate another equivalence relation on a vertex set of a connected graph. First, we need some additional definitions and notation. Let $A = \left\{\{a_1, a_2, \ldots, a_k\} \mid a_i \in \mathbb{Z} \text{ for } 1 \le i \le k\right\}$ be a collection of multisets of integers. For an integer c, we define

$$\{a_1, a_2, \ldots, a_k\} + \{c, c, \ldots, c\} = \{a_1 + c, a_2 + c, \ldots, a_k + c\},$$

where $\{a_1, a_2, \ldots, a_k\}, \{c, c, \ldots, c\} \in A$. Let W be a set of vertices of a connected graph G and let u and v be vertices of G. A *multisimilar relation* R_W with respect to W on a vertex set $V(G)$ is defined by $u \; R_W \; v$ if there is an integer $c_W(u,v)$ such that

$$mr(u \mid W) = mr(v \mid W) + \{c_W(u,v), c_W(u,v), \ldots, c_W(u,v)\}. \qquad (3.3.1)$$

An integer $c_W(u,v)$ satisfying (3.3.1) is called a *multisimilar constant of* $u\ R_W\ v$ or simply a *multisimilar constant*. Clearly, R_W is an equivalence relation on $V(G)$. For each vertex u in $V(G)$, let $[u]_W$ denote the *multisimilar class* of u with respect to W. Then

$$x \in [u]_W \text{ if and only if}$$
$$mr(x \mid W) = mr(u \mid W) + \{c_W(x,u), c_W(x,u), \ldots, c_W(x,u)\}, \tag{3.3.2}$$

where $c_W(x,u)$ is a multisimilar constant. Observe that if $x \in [u]_W$, then there is a multisimilar constant $c_W(x,u)$ with a property that, for every vertex $w \in W$, there is a corresponding vertex $w' \in W$ such that

$$d(x,w) = d(u,w') + c_W(x,u). \tag{3.3.3}$$

With this observation, we may as well say that $x \in [u]_W$ if and only if there are multisimilar constant $c_W(x,u)$ and a bijective function f on W defined as $f(w) = w'$ whenever $d(x,w) = d(u,w') + c_W(x,u)$. This function is called a *multisimilar function of* $x\ R_W\ u$ or a *multisimilar function* if there is no ambiguity. Consequently, it is not surprising that an inverse function f^{-1} is also a multisimilar function of $u\ R_W\ x$ with a multisimilar constant $c_W(u,x) = -c_W(x,u)$. To illustrate this concept, let us consider the set $W = \{w,x,y\}$ of the graph G of Figure 13.

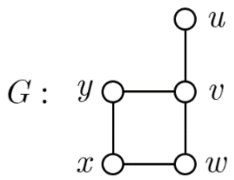

Figure 13: The graph G

The multirepresentations of vertices of G with respect to W are

$$mr(u \mid W) = \{2,2,3\}, \quad mr(v \mid W) = \{1,1,2\}, \quad mr(w \mid W) = \{0,1,2\},$$
$$mr(x \mid W) = \{0,1,1\}, \quad mr(y \mid W) = \{0,1,2\}.$$

Since $mr(u \mid W) = mr(v \mid W) + \{1,1,1\}$, $mr(w \mid W) = mr(y \mid W) + \{0,0,0\}$ and $mr(x \mid W) = mr(x \mid W) + \{0,0,0\}$, it follows that $u\ R_W\ v$ with $c_W(u,v) = 1$, $w\ R_W\ y$ with $c_W(w,y) = 0$ and $x\ R_W\ x$ with $c_W(x,x) = 0$, respectively. In fact, $[u]_W = \{u,v\}$, $[w]_W = \{w,y\}$ and $[x]_W = \{x\}$. By considering a multisimilar class $[u]_W$, a multisimilar function f of $u\ R_W\ v$ is defined by $f(w) = w$, $f(x) = x$ and $f(y) = y$. Moreover, there is another multisimilar function f' of $u\ R_W\ v$, that is, $f'(w) = y$, $f'(x) = x$ and $f'(y) = w$.

The example just described shows an important point that a multisimilar function of any two vertices in the same multisimilar class with respect to a set W is not necessarily unique.

More generally, for a vertex u and a set W of vertices of a connected graph G, let $mr(u \mid W) = \{r_1 \cdot a_1, r_2 \cdot a_2, \ldots, r_l \cdot a_l\}$, where $a_1 < a_2 < \cdots < a_l$ and r_i is a repetition number of type a_i for each i with $1 \le i \le l$. Assume that there is a vertex v of G belonging to the same multisimilar class as u, that is, $v \in [u]_W$. By (3.3.2) and (3.3.3), for each type of $mr(u \mid W)$, there is a corresponding type of $mr(v \mid W)$ such that their repetition numbers are equal. Therefore, we may assume that $mr(v \mid W) = \{r_1 \cdot b_1, r_2 \cdot b_2, \ldots, r_l \cdot b_l\}$, where $b_1 < b_2 < \cdots < b_l$. For each integer i with $1 \le i \le l$, let $A_i = \{w \in W \mid d(u,w) = a_i\}$ and $B_i = \{w \in W \mid d(v,w) = b_i\}$. Then the types of $mr(u \mid W)$ partition W into l sets $A_1, A_2, \ldots, A_l$. On the other hand, W is also partitioned into l sets $B_1, B_2, \ldots, B_l$ depending on the types of $mr(v \mid W)$. Hence, the multisimilar function f of $u\ R_W\ v$ has the property that, for every vertex $w \in A_i$, there is a vertex $w' \in B_i$ such that $f(w) = w'$, where $1 \le i \le l$. Indeed, there are $r_1!\,r_2!\cdots r_l!$ distinct multisimilar functions of $u\ R_W\ v$. These observations yield the following result.

Theorem 3.3.1. *Let W be a set of vertices of a connected graph G and let u and v be vertices of G such that $u \in [v]_W$. Suppose that $mr(u \mid W) = \{r_1 \cdot a_1, r_2 \cdot a_2, \ldots, r_l \cdot a_l\}$, where $a_1 < a_2 < \cdots < a_l$ and r_i is a repetition number of type a_i for each integer i with $1 \le i \le l$. Then*

(i) $mr(v \mid W) = \{r_1 \cdot b_1, r_2 \cdot b_2, \ldots, r_l \cdot b_l\}$ *for some integers* $b_1, b_2, \ldots, b_l$ *with* $b_1 < b_2 < \cdots < b_l$,

(ii) *there is a multisimilar function* f *of* $u\ R_W\ v$ *such that* $f(w_i) = w'_i$, *where* $d(u,w_i) = a_i$ *and* $d(v,w'_i) = b_i$ *for each* i *with* $1 \le i \le l$,

(iii) *there are* $r_1!r_2!\cdots r_l!$ *distinct multisimilar functions of* $u\ R_W\ v$.

By Theorem 3.3.1, the following result is obtained.

Corollary 3.3.2. *Let* W *be a set of vertices of a connected graph* G *and let* u *and* v *be vertices of* G *such that* $u \in [v]_W$ *with a multisimilar constant* $c_W(u,v)$. *Then*

(i) *if* M_1 *and* M_2 *are the maximum elements of* $mr(u \mid W)$ *and* $mr(v \mid W)$, *respectively, then* $M_1 = M_2 + c_W(u,v)$,

(ii) *if* m_1 *and* m_2 *are the minimum elements of* $mr(u \mid W)$ *and* $mr(v \mid W)$, *respectively, then* $m_1 = m_2 + c_W(u,v)$.

Proof. Suppose that $u \in [v]_W$ and $|W| = l$. Let $mr(u \mid W) = \{r_1 \cdot a_1, r_2 \cdot a_2, \ldots, r_l \cdot a_l\}$ and $mr(v \mid W) = \{r_1 \cdot b_1, r_2 \cdot b_2, \ldots, r_l \cdot b_l\}$, where $a_1 < a_2 < \cdots < a_l$ and $b_1 < b_2 < \cdots < b_l$. Since M_1 and M_2 are the maximum elements of $mr(u \mid W)$ and $mr(v \mid W)$, respectively, it follows that there are vertices w and w' in W such that $M_1 = d(u,w) = a_l$ and $M_2 = d(v,w') = b_l$. By Theorem 3.3.1, there is a multisimilar function f of $u\ R_W\ v$ such that $f(w) = w'$. Then $d(u,w) = d(v,w') + c_W(u,v)$, where $c_W(u,v)$ is a multisimilar constant. Thus, (i) holds. For (ii), the statement may be proven in the same way as (i), and therefore such proof is omitted.

Next, we are prepared to establish the upper bound for the cardinality of a multisimilar class of a vertex in a connected graph. To show this, let us present a useful proposition as follows.

Proposition 3.3.3. *Let* W *be a set of vertices of a connected graph* G *and let* u *and* v *be vertices of* G *such that* $u \in [v]_W$. *Then* $mr(u \mid W)$ *and* $mr(v \mid W)$ *have the same minimum (or maximum) element if and only if* $mr(u \mid W) = mr(v \mid W)$.

Proof. If $mr(u \mid W) = mr(v \mid W)$, then the minimum (and maximum) elements of $mr(u \mid W)$ and $mr(v \mid W)$ are the same. For the converse, assume that m_1 and m_2 are the minimum elements of $mr(u \mid W)$ and $mr(v \mid W)$, respectively, such that $m_1 = m_2$. Since $u \in [v]_W$, there is a multisimilar constant $c_W(u,v)$ such that $mr(u \mid W) = mr(v \mid W) + \{c_W(u,v), c_W(u,v), \ldots, c_W(u,v)\}$. By Corollary 3.3.2 (ii), it follows that $m_1 = m_2 + c_W(u,w)$. Thus, $c_W(u,v) = 0$. Hence, $mr(u \mid W) = mr(v \mid W)$. Similarly, if $mr(u \mid W)$ and $mr(v \mid W)$ have the same maximum element, then $mr(u \mid W) = mr(v \mid W)$.

Theorem 3.3.4. *If W is a multiresolving set of a connected graph G, then the cardinality of multisimilar class of each vertex of G with respect to W is at most $\mathrm{diam}(G)+1$.*

Proof. Assume, to the contrary, that there is a vertex v of G such that $[v]_W$ has the cardinality at least $\mathrm{diam}(G)+2$. Since the minimum elements of multirepresentations of vertices in $[v]_W$ with respect to W have at most $\mathrm{diam}(G)+1$ distinct values, there are at least two vertices x and y in $[v]_W$ having the same value of the minimum element of $mr(x \mid W)$ and $mr(y \mid W)$. Therefore, $mr(x \mid W) = mr(y \mid W)$ by Proposition 3.3.3, contradicting the fact that W is a multiresolving set of G.

We can show that the upper bound in Theorem 3.3.4 is sharp by considering the path $P_n = (v_1, v_2, \ldots, v_n)$. We have that $\mathrm{diam}(P_n) = n-1$ and a set $W = \{v_1\}$ is a multiresolving set of P_n. Thus, $[v_1]_W$ contains all vertices of P_n and so $\left|[v_1]_W\right| = n$.

The next result describes the properties of multisimilar classes with respect to a set of vertices.

Theorem 3.3.5. *Let u and v be vertices of a connected graph G and let W be a set of vertices of G. Then*

(i) *if $[u]_W \neq [v]_W$, then $mr(x \mid W) \neq mr(y \mid W)$ for all $x \in [u]_W$ and $y \in [v]_W$,*

(ii) *if $[u]_W = \{u\}$ for all $u \in V(G)$, then W is a multiresolving set of G.*

Proof. (i) Assume, to the contrary, that there exist two distinct vertices $x \in [u]_W$ and $y \in [v]_W$ such that $mr(x \mid W) = mr(y \mid W)$. Then there are multisimilar constants $c_W(x,u)$ and $c_W(y,v)$ such that

$$mr(x \mid W) = mr(u \mid W) + \{c_W(x,u), c_W(x,u), \ldots, c_W(x,u)\} \text{ and}$$
$$mr(y \mid W) = mr(v \mid W) + \{c_W(y,v), c_W(y,v), \ldots, c_W(y,v)\}.$$

Therefore, $mr(u \mid W) + \{c_W(x,u),\ldots,c_W(x,u)\} = mr(v \mid W) + \{c_W(y,v),\ldots,c_W(y,v)\}$. Thus, $mr(u \mid W) = mr(v \mid W) + \{c_W(y,v) - c_W(x,u),\ldots,c_W(y,v) - c_W(x,u)\}$. Hence, u belongs to $[v]_W$, which is a contradiction.

(ii) Assume, to the contrary, that W is not a multiresolving set of G. Then there exist two distinct vertices x and y such that $mr(x \mid W) = mr(y \mid W)$. Hence, y belongs to $[x]_W$, producing a contradiction.

3.4 The characterization of caterpillars with multidimension 3

A *caterpillar* is a tree of order at least 3, the removal of whose end-vertices produces a path called the *spine* of the caterpillar. A vertex of the spine of the caterpillar is called a *spine-vertex*. Let T be a caterpillar that $\dim_M(T)$ is defined. Since any two end-vertices that are adjacent to the same spine-vertex of T are distance-similar, it follows by Theorem I that there are at most two end-vertices that are adjacent to each spine-vertex of T. Therefore, we consider multiresolving sets of such a caterpillar. In order to do this, let us introduce some additional definitions and notation. For integers $s,k_1,k_2,\ldots,k_s$ with $s \geq 1$, $1 \leq k_1,k_s \leq 2$ and $0 \leq k_2,k_3,\ldots,k_{s-1} \leq 2$, let $\mathrm{ca}(k_1,k_2,\ldots,k_s)$ be a caterpillar which is obtained from the spine $(u_1,u_2,\ldots,u_s)$ by joining k_i end-vertices to the spine vertex u_i, where $1 \leq i \leq s$. Observe that, if $k_i = 0$, then there is no end-vertices joining to the spine vertex u_i. Also, if $k_i = 1$, then the spine-vertex u_i is adjacent to an end-vertex which is called the *first end-vertex* of u_i and denoted by v_i. Furthermore, if $k_i = 2$, then there are two end-vertices joining to u_i that are called the *first* and *second end-vertices* of u_i and denoted by v_i and w_i, respectively. For each integer i with $1 \leq i \leq s$, we define a set $\Psi = \{i \in \mathbb{Z} \mid k_i = 2\}$ to be the *second end-set* of a caterpillar $\mathrm{ca}(k_1,k_2,\ldots,k_s)$. To emphasize that this is the second end-set Ψ of a caterpillar T, we sometimes denote this set by Ψ_T. For example, the caterpillar $\mathrm{ca}(1,2,0,2,0,2)$ of Figure 14 has six spine-vertices, namely, u_1,u_2,u_3,u_4,u_5,u_6. Since no end-vertex is adjacent to a spine vertex u_3, as well as to a spine vertex u_5, it follows that there are no first and second end-vertices of u_3 and u_5. The first end-vertices of u_1,u_2,u_4 and u_6 are v_1,v_2,v_4 and v_6, respectively. Also, the second end-vertices of

u_2, u_4 and u_6 are w_2, w_4 and w_6, respectively. Therefore, the second end-set of $\mathrm{ca}(1,2,0,2,0,2)$ is the set $\Psi = \{2,4,6\}$.

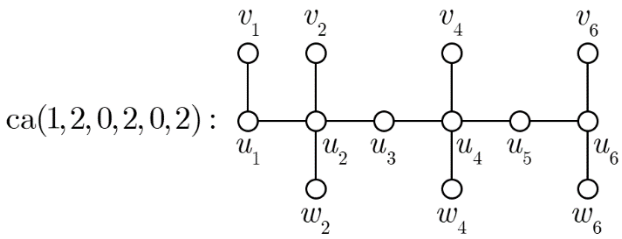

Figure 14: The caterpillar $\mathrm{ca}(1,2,0,2,0,2)$ with the second end-set $\Psi = \{2,4,6\}$

The following observation is a consequence of Theorem I.

Observation 3.4.1. *Every multiresolving set of a caterpillar* $\mathrm{ca}(k_1, k_2, \ldots, k_s)$ *with a second end-set* Ψ *contains either first end-vertex* v_i *or second end-vertex* w_i*, where* $i \in \Psi$.

Proof. For each integer $i \in \Psi$, since v_i and w_i are distance-similar, it follows by Theorem I that every multiresolving set of $\mathrm{ca}(k_1, k_2, \ldots, k_s)$ contains exactly one of $\{v_i, w_i\}$.

Next, we are prepared to characterize caterpillars having multidimension 3. In order to do this, we first present several preliminary results.

Proposition 3.4.2. *Let* s, α, β *be integers with* $s \geq 3$ *and* $1 \leq \alpha < \beta \leq s$*, and let* W *be a set of vertices of a caterpillar* $\mathrm{ca}(k_1, k_2, \ldots, k_s)$ *containing one of* $\{v_1, w_1\}$ *and one of* $\{v_s, w_s\}$*. If* $mr(u_\alpha \mid W) = mr(u_\beta \mid W)$ *or* $mr(v_\alpha \mid W) = mr(v_\beta \mid W)$*, then* $1 \leq \alpha \leq \left\lceil \frac{s}{2} \right\rceil$ *and* $\beta = s - \alpha + 1$.

Proof. Suppose that $mr(u_\alpha \mid W) = mr(u_\beta \mid W)$. Without loss of generality, assume that W contains v_1 and v_s. For $1 \leq \alpha < \beta \leq \left\lceil \frac{s}{2} \right\rceil$, since $d(u_\alpha, v_s) = s - \alpha + 1$ and $d(u_\beta, v_s) = s - \beta + 1$ are the maximum elements of $mr(u_\alpha \mid W)$ and $mr(u_\beta \mid W)$, respectively, it follows that $\alpha = \beta$, which is a contradiction. For $\left\lceil \frac{s}{2} \right\rceil + 1 \leq \alpha < \beta \leq s$, since $d(u_\alpha, v_1) = \alpha$ and $d(u_\beta, v_1) = \beta$ are the maximum elements of $mr(u_\alpha \mid W)$ and $mr(u_\beta \mid W)$, respectively, it follows that $\alpha = \beta$, a contradiction is produced. Thus,

$1 \le \alpha \le \left\lceil \frac{s}{2} \right\rceil$ and $\left\lceil \frac{s}{2} \right\rceil + 1 \le \beta \le s$. Moreover, since $d(u_\alpha, v_s) = s - \alpha + 1$ and $d(u_\beta, v_1) = \beta$ are the maximum elements of $mr(u_\alpha \mid W)$ and $mr(u_\beta \mid W)$, respectively, it follows that $\beta = s - \alpha + 1$, as we claimed. If $mr(v_\alpha \mid W) = mr(v_\beta \mid W)$, then it can be obtained in a similar manner.

Proposition 3.4.3. *Let s, γ, δ be integers with $s \ge 3$ and $1 \le \gamma, \delta \le s$, and let W be a set of vertices of a caterpillar $\mathrm{ca}(k_1, k_2, \ldots, k_s)$ containing one of $\{v_1, w_1\}$ and one of $\{v_s, w_s\}$. Then*

(i) *if $1 \le \gamma < \delta \le s$ and $mr(v_\gamma \mid W) = mr(u_\delta \mid W)$, then $1 \le \gamma \le \left\lceil \frac{s}{2} \right\rceil$ and $\delta = s - \gamma + 2$, and*

(ii) *if $1 \le \delta \le \gamma \le s$ and $mr(v_\gamma \mid W) = mr(u_\delta \mid W)$, then $\left\lceil \frac{s}{2} \right\rceil + 1 \le \gamma \le s$ and $\delta = s - \gamma$.*

Proof. (i) Suppose that $1 \le \gamma < \delta \le s$ and $mr(v_\gamma \mid W) = mr(u_\delta \mid W)$. Without loss of generality, let us assume that W contains v_1 and v_s. If $1 \le \gamma < \delta \le \left\lceil \frac{s}{2} \right\rceil$, then $d(v_\gamma, v_s) = s - \gamma + 2$ and $d(u_\delta, v_s) = s - \delta + 1$ are the maximum elements of $mr(v_\gamma \mid W)$ and $mr(u_\delta \mid W)$, respectively. Therefore, $\delta = \gamma - 1$, that is, $\gamma > \delta$, which gives a contradiction. If $\left\lceil \frac{s}{2} \right\rceil + 1 \le \gamma < \delta \le s$, then $d(v_\gamma, v_1) = \gamma + 1$ and $d(u_\delta, v_1) = \delta$ are the maximum elements of $mr(v_\gamma \mid W)$ and $mr(u_\delta \mid W)$, respectively. Thus, $\delta = \gamma + 1$. Since $d(v_\gamma, v_s) = s - \gamma + 2$ belongs to $mr(v_\gamma \mid W)$, there is a vertex w for which $w = u_{2\delta-s-3}$ or $w = v_{2\delta-s-2}$ or $w = w_{2\delta-s-2}$ such that $d(u_\delta, w) = s - \gamma + 2$. Moreover, since $d(v_\gamma, w) = d(u_\delta, w) = s - \gamma + 2$, it follows that $mr(v_\gamma \mid W)$ contains $s - \gamma + 2$'s more than $mr(u_\delta \mid W)$ does, which is impossible. Therefore, $1 \le \gamma \le \left\lceil \frac{s}{2} \right\rceil$ and $\left\lceil \frac{s}{2} \right\rceil + 1 \le \delta \le s$. Moreover, since $d(v_\gamma, v_s) = s - \gamma + 2$ and $d(u_\delta, v_1) = \delta$ are the maximum elements of $mr(v_\gamma \mid W)$ and $mr(u_\delta \mid W)$, respectively, it follows that

$\delta = s - \gamma + 2$, as we claimed. For (ii), the statement may be proven in the same way as (i), and therefore such proof is omitted.

An argument similar to the one used in the proof of Propositions 3.4.2 and 3.4.3 establishes the following results.

Proposition 3.4.4. *Let* s, α, β *be integers with* $s \geq 3$ *and* $1 \leq \alpha < \beta \leq s$ *, and let* W *be a set of vertices of a caterpillar* $\mathrm{ca}(k_1, k_2, \ldots, k_s)$ *containing* u_1 *and one of* $\{v_s, w_s\}$ *except* v_1 *and* w_1*. If* $mr(u_\alpha \mid W) = mr(u_\beta \mid W)$ *or* $mr(v_\alpha \mid W) = mr(v_\beta \mid W)$ *, then* $1 \leq \alpha \leq \left\lceil \frac{s}{2} \right\rceil$ *and* $\beta = s - \alpha + 2$.

Proposition 3.4.5. *Let* s, γ, δ *be integers with* $s \geq 3$ *and* $1 \leq \gamma, \delta \leq s$ *, and let* W *be a set of vertices of a caterpillar* $\mathrm{ca}(k_1, k_2, \ldots, k_s)$ *containing* u_1 *and one of* $\{v_s, w_s\}$ *except* v_1 *and* w_1*. Then*

(i) *if* $1 \leq \gamma < \delta \leq s$ *and* $mr(v_\gamma \mid W) = mr(u_\delta \mid W)$*, then* $1 \leq \gamma \leq \left\lceil \frac{s}{2} \right\rceil$ *and* $\delta = s - \gamma + 3$*, and*

(ii) *if* $1 \leq \delta \leq \gamma \leq s$ *and* $mr(v_\gamma \mid W) = mr(u_\delta \mid W)$*, then* $\left\lceil \frac{s}{2} \right\rceil + 1 \leq \gamma \leq s$ *and* $\delta = s - \gamma + 1$.

We now establish a characterization of a caterpillar $\mathrm{ca}(k_1, k_2, \ldots, k_s)$ with multidimension 3. For $s = 1$ and $s = 2$, the caterpillars $\mathrm{ca}(k_1)$ and $\mathrm{ca}(k_1, k_2)$ are shown in Figure 15, where the vertices of a multibasis of these caterpillars are indicated by solid vertices.

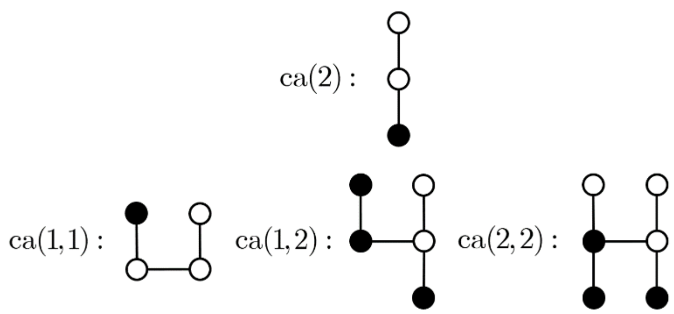

Figure 15: The caterpillars $\mathrm{ca}(2)$, $\mathrm{ca}(1,1)$, $\mathrm{ca}(1,2)$ and $\mathrm{ca}(2,2)$

Notice that $\mathrm{ca}(2) = P_3$, $\mathrm{ca}(1,1) = P_4$ and $\mathrm{ca}(1,2) = \mathrm{ca}(2,1)$. This implies that there is no caterpillar having multidimension 3, where $s = 1$, and there are two distinct caterpillars having multidimension 3, where $s = 2$. For $s = 3$, it is routine to verify that $\mathrm{ca}(1,0,2) = \mathrm{ca}(2,0,1)$, $\mathrm{ca}(1,1,1)$, $\mathrm{ca}(1,1,2) = \mathrm{ca}(2,1,1)$, $\mathrm{ca}(2,0,2)$ and $\mathrm{ca}(2,1,2)$ are caterpillars having multidimension 3. For $s \geq 4$, let us introduce some additional definitions and notation.

For an even integer $s \geq 4$, let T_1 be a caterpillar $\mathrm{ca}(k_1, k_2, \ldots, k_s)$ such that $\Psi = \{1, r, s\}$, where $r \in \{2, 3, \ldots, s-1\}$. In particular, for $s = 8$ and $r = 3$, the caterpillar $T_1 = \mathrm{ca}(2,0,2,1,0,1,0,2)$ with $\Psi = \{1,3,8\}$ is shown in Figure 16.

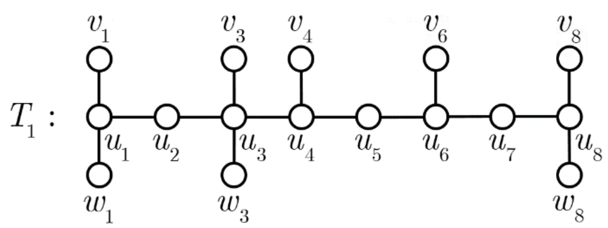

Figure 16: The caterpillar $T_1 = \mathrm{ca}(2,0,2,1,0,1,0,2)$ with $\Psi = \{1,3,8\}$

For an odd integer $s \geq 5$, let T_2 be a caterpillar $\mathrm{ca}(k_1, k_2, \ldots, k_s)$ such that $\Psi = \{1, r, s\}$, where

$$r \in \begin{cases} \{2,3,..,s-1\} - \{3, \frac{s+1}{2}, s-2\} & \text{if } s \equiv 1 \pmod 4, \\ \{2,3,..,s-1\} - \{3, \frac{s-1}{2}, \frac{s+1}{2}, \frac{s+3}{2}, s-2\} & \text{if } s \equiv 3 \pmod 4. \end{cases} \tag{3.4.1}$$

For example, for $s = 9$ and $r = 4$, the caterpillar $T_2 = \mathrm{ca}(2,0,1,2,0,1,1,0,2)$ with $\Psi = \{1,4,9\}$ is illustrated in Figure 17.

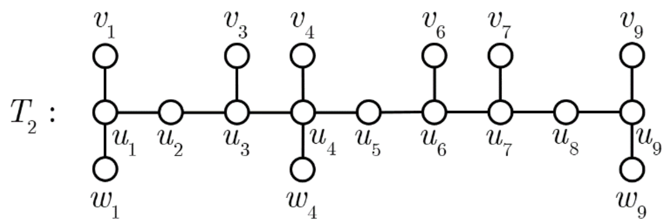

Figure 17: The caterpillar $T_2 = \mathrm{ca}(2,0,1,2,0,1,1,0,2)$ with $\Psi = \{1,4,9\}$

For an odd integer $s \geq 9$, let T_3 be a caterpillar $\mathrm{ca}(k_1, k_2, ..., k_s)$ such that either $\Psi = \{1, 3, s\}$ and $k_{\frac{s-1}{2}} = 0$, or $\Psi = \{1, s-2, s\}$ and $k_{\frac{s+3}{2}} = 0$. For an odd integer $s \geq 11$ and $s \equiv 3 \pmod 4$, let T_4 be a caterpillar $\mathrm{ca}(k_1, k_2, ..., k_s)$ such that either $\Psi = \{1, \frac{s-1}{2}, s\}$ and $k_{\frac{s+5}{4}} = 0$, or $\Psi = \{1, \frac{s+3}{2}, s\}$ and $k_{\frac{3s-1}{4}} = 0$.

Proposition 3.4.6. *A caterpillar T_i, where $1 \leq i \leq 4$ has multidimension 3.*

Proof. For each integer i with $1 \leq i \leq 4$, we show that every caterpillar T_i has multidimension 3. We verify this for T_2 only since the proof for T_1, T_3 and T_4 uses an argument similar to the one for T_2. First, we verify that $W = \{w_1, w_r, w_s\}$ is a multiresolving set of T_2, where r satisfies the condition (3.4.1). Without loss of generality, we may assume that $2 \leq r \leq \left\lceil \frac{s}{2} \right\rceil$. The multirepresentations of vertices of W with respect to W are $mr(w_1 \mid W) = \{0, r+1, s+1\}, mr(w_r \mid W) = \{0, r+1, s-r+2\}$ and $mr(w_s \mid W) = \{0, s-r+2, s+1\}$. Since $r \notin \{1, \frac{s+1}{2}, s\}$, it follows that these 3-multisets are distinct. Next, we claim that $mr(x \mid W) \neq mr(y \mid W)$ for all vertices $x, y \in V(T_2) - W$. Suppose, contrary to our claim, that $mr(x \mid W) = mr(y \mid W)$ for some vertices $x, y \in V(T_2) - W$. We consider three cases.

Case 1. x and y are spine-vertices.

Let $x = u_\alpha$ and $y = u_\beta$, where $1 \leq \alpha < \beta \leq s$. Then by Proposition 3.4.2, we obtain that $1 \leq \alpha \leq \left\lceil \frac{s}{2} \right\rceil$ and $\beta = s - \alpha + 1$. Thus, $mr(u_\beta \mid W) = \{s - \beta + 1, \beta - r + 1, \beta\} = \{\alpha, s - \alpha - r + 2, s - \alpha + 1\}$. Since $mr(u_\alpha \mid W) = \{\alpha, |\alpha - r| + 1, s - \alpha + 1\}$, it follows that $|\alpha - r| + 1 = s - \alpha - r + 2$. If $\alpha \geq r$, then $2\alpha = s + 1$ and so $\alpha = \beta$, which is impossible. If $\alpha < r$, then $r = \frac{s-1}{2}$, a contradiction.

Case 2. x and y are first end-vertices.

Let $x = v_\alpha$ and $y = v_\beta$, where $1 \leq \alpha < \beta \leq s$. Proposition 3.4.2 implies that $1 \leq \alpha \leq \left\lceil \frac{s}{2} \right\rceil$ and $\beta = s - \alpha + 1$. Thus, $mr(v_\beta \mid W) = \{s - \beta + 2, \beta - r + 2, \beta + 1\} = \{\alpha + 1, s - \alpha - r + 3, s - \alpha + 2\}$. Since $mr(v_\alpha \mid W) = \{\alpha + 1, |\alpha - r| + 2, s - \alpha + 2\}$, it follows that $|\alpha - r| + 2 = s - \alpha - r + 3$. If $\alpha \geq r$, then $2\alpha = s + 1$ and so $\alpha = \beta$, which cannot occur. If $\alpha < r$, then $r = \dfrac{s-1}{2}$. This is also a contradiction.

Case 3. x is a first end-vertex and y is a spine-vertex.

Let $x = v_\gamma$ and $y = u_\delta$, where $1 \leq \gamma, \delta \leq s$. We consider two subcases.

Subcase 3.1. $1 \leq \gamma < \delta \leq s$.

Then by Proposition 3.4.3 (i), we obtain that $1 \leq \gamma \leq \left\lceil \dfrac{s}{2} \right\rceil$ and $\delta = s - \gamma + 2$. Thus, $mr(u_\delta \mid W) = \{s - \delta + 1, \delta - r + 1, \delta\} = \{\gamma - 1, s - \gamma - r + 3, s - \gamma + 2\}$. Since $mr(v_\gamma \mid W) = \{\gamma + 1, |\gamma - r| + 2, s - \gamma + 2\}$, it follows that $|\gamma - r| + 2 = \gamma - 1$ and $\gamma + 1 = s - \gamma - r + 3$. If $\gamma \geq r$, then $r = 3$, which is impossible. If $\gamma < r$, then $s = 4(\gamma - 2) + 3$, that is, $s \equiv 3 (\bmod 4)$. Also, we obtain that $2r = s - 1$, and then $r = \dfrac{s-1}{2}$, which is a contradiction.

Subcase 3.2. $1 \leq \delta \leq \gamma \leq s$.

Therefore, by Proposition 3.4.3 (ii), we obtain that $\left\lceil \dfrac{s}{2} \right\rceil + 1 \leq \gamma \leq s$ and $\delta = s - \gamma$. Thus, $mr(v_\gamma \mid W) = \{s - \gamma + 2, \gamma - r + 2, \gamma + 1\} = \{\delta + 2, s - \delta - r + 2, s - \delta + 1\}$. Since $mr(u_\delta \mid W) = \{\delta, |\delta - r| + 1, s - \delta + 1\}$, it follows that $|\delta - r| + 1 = \delta + 2$ and $\delta = s - \delta - r + 2$. Consequently, $|\delta - r| = s - \delta - r + 3$. If $\delta \geq r$, then $2\delta = s + 3$, which cannot occur. If $\delta < r$, then $2r = s + 3$, a contradiction.

Therefore, $mr(x \mid W) \neq mr(y \mid W)$ for all vertices $x, y \in V(T_2) - W$, that is, W is a multiresolving set of T_2 and so $\dim_M(T_2) \leq 3$. Since T_2 is not a path, it follows by Theorems J and K that $\dim_M(T_2) \geq 3$. Hence, $\dim_M(T_2) = 3$.

The following corollary is an immediate consequence of Proposition 3.4.6.

Corollary 3.4.7. *If T is a caterpillar $\mathrm{ca}(k_1, k_2, \ldots, k_s)$ such that $T = T_i$, where $1 \leq i \leq 4$ with $\Psi = \{1, r, s\}$, then $\{x_1, x_r, x_s\}$ is a multibasis of T, where $x_i \in \{v_i, w_i\}$ for $i = 1, r, s$.*

For an integer $s \geq 4$, let T_5 be a caterpillar $\mathrm{ca}(k_1, k_2, \ldots, k_s)$ such that either $\Psi = \{p, s\}$ or $\Psi = \{1, q\}$, where $1 \leq p < q \leq s$.

Proposition 3.4.8. *A caterpillar T_5 has multidimension 3.*

Proof. First, suppose that $\Psi = \{p, s\}$, where $1 \le p \le s-1$. Since T_5 is not a path, it follows by Theorems J and K that $\dim_M(T_5) \ge 3$. Next, we consider two cases.

Case 1. $p = 1$.

We show that $W = \{u_1, w_1, w_s\}$ is a multiresolving set of T_5. The multirepresentations of vertices of W with respect to W are $mr(u_1 \mid W) = \{0,1,s\}, mr(w_1 \mid W) = \{0,1,s+1\}$ and $mr(w_s \mid W) = \{0,s,s+1\}$. Thus, these $3-$multisets are distinct. Next, we claim that $mr(x \mid W) \ne mr(y \mid W)$ for all vertices $x, y \in V(T_5) - W$. Assume, to the contrary, that $mr(x \mid W) = mr(y \mid W)$ for some vertices $x, y \in V(T_5) - W$. We consider three subcases.

Subcase 1.1. x and y are spine-vertices.

Let $x = u_\alpha$ and $y = u_\beta$, where $1 \le \alpha < \beta \le s$. Then by Proposition 3.4.2, $1 \le \alpha \le \left\lceil \frac{s}{2} \right\rceil$ and $\beta = s - \alpha + 1$. Thus, $mr(u_\beta \mid W) = \{s-\beta+1, \beta-1, \beta\} = \{\alpha$ $s-\alpha, s-\alpha+1\}$. Since $mr(u_\alpha \mid W) = \{\alpha, \alpha-1, s-\alpha+1\}$, it follows that $\alpha - 1 =$ $s - \alpha$ and so $\alpha = \beta$, which is impossible.

Subcase 1.2. x and y are first end-vertices.

Let $x = v_\alpha$ and $y = v_\beta$, where $1 \le \alpha < \beta \le s$. Then by Proposition 3.4.2, we have that $1 \le \alpha \le \left\lceil \frac{s}{2} \right\rceil$ and $\beta = s - \alpha + 1$. Thus, $mr(v_\beta \mid W) = \{s-\beta+2, \beta, \beta+1\}$ $= \{\alpha+1, s-\alpha+1, s-\alpha+2\}$. Since $mr(v_\alpha \mid W) = \{\alpha+1, \alpha, s-\alpha+2\}$, it follows that $\alpha = s - \alpha + 1$ and so $\alpha = \beta$. This is a contradiction.

Subcase 1.3. x is a first end-vertex and y is a spine-vertex.

Let $x = v_\gamma$ and $y = u_\delta$, where $1 \le \gamma, \delta \le s$. We consider two subcases.

Subcase 1.3.1. $1 \le \gamma < \delta \le s$.

Then by Proposition 3.4.3 (i), we obtain that $1 \le \gamma \le \left\lceil \frac{s}{2} \right\rceil$ and $\delta = s - \gamma + 2$.

Since m (${}_\delta r$ | ≠) δW { δ s δ , $\gamma \ne 1$, − ; and $mr(v_\gamma \mid W) = \{\gamma+1, \gamma, s-\gamma+2\}$, it follows that $mr(v_\gamma \mid W) \ne mr(u_\delta \mid W)$, which is impossible.

Subcase 1.3.2. $1 \le \delta \le \gamma \le s$.

Then by Proposition 3.4.3 (ii), $\left\lceil \frac{s}{2} \right\rceil + 1 \le \gamma \le s$ and $\delta = s - \gamma$. Since $mr(v_\gamma \mid W) = \{s-\gamma+2, \gamma, \gamma+1\} = \{\delta+2, s-\delta, s-\delta+1\}$ and $mr(u_\delta \mid W) = \{\delta, \delta-1, s-\delta+1\}$, it follows that $mr(v_\gamma \mid W) \ne mr(u_\delta \mid W)$, this is also contradiction. Therefore, $mr(x \mid W) \ne mr(y \mid W)$ for all vertices $x, y \in V(T_5) - W$, that is, W is a multiresolving set of T_5. Hence, $\dim_M(T_5) \le 3$ and so $\dim_M(T_5) = 3$, where $p = 1$.

Case 2. $p \ge 2$.

We consider two subcases.

Subcase 2.1. s is even.

With the aid of Theorem 3.2.1 and Corollary 3.4.7, since $T_5 = T_1 - w_1$ and $W = \{v_1, w_p, w_s\}$ is a multiresolving set of T_1, it follows that W is a multiresolving set of T_5. Therefore, $\dim_M(T_5) \le 3$. Hence, $\dim_M(T_5) = 3$, where $p \ge 2$ and s is even.

Subcase 2.2. s is odd.

We consider two subcases.

Subcase 2.2.1. $p = 2$.

By Theorem 3.2.1 and Corollary 3.4.7, since $T_5 = T_2 - w_1$ and $W = \{v_1, w_p, w_s\}$ is a multiresolving set of T_2, it follows that W is a multiresolving set of T_5. Therefore, $\dim_M(T_5) \le 3$. Hence, $\dim_M(T_5) = 3$, where $p = 2$ and s is odd.

Subcase 2.2.2. $p \ge 3$.

We show that the set $W = \{u_1, w_p, w_s\}$ is a multiresolving set of T_5. The multirepresentations of vertices of W with respect to W are $mr(u_1 \mid W) = \{0, p, s\}, mr(w_p \mid W) = \{0, p, s-p+2\}$ and $mr(w_s \mid W) = \{0, s-p+2, s\}$. Thus, these 3-multisets are distinct. Next, we claim that $mr(x \mid W) \ne mr(y \mid W)$ for all vertices $x, y \in V(T_5) - W$. Assume, to the contrary, that $mr(x \mid W) = mr(y \mid W)$ for some vertices $x, y \in V(T_5) - W$. We consider three subcases.

Subcase 2.2.2.1. x and y are spine-vertices.

Let $x = u_\alpha$ and $y = u_\beta$, where $1 \le \alpha < \beta \le s$. Then by Proposition 3.4.4, $1 \le \alpha \le \left\lceil \frac{s}{2} \right\rceil$ and $\beta = s - \alpha + 2$. Thus, $mr(u_\beta \mid W) = \{s-\beta+1, |\beta - p| + 1, \beta - 1\}$

$= \{\alpha - 1, |\beta - p| + 1, s - \alpha + 1\}$. Since $mr(u_\alpha \mid W) = \{\alpha - 1, |\alpha - p| + 1, s - \alpha + 1\}$, it follows that $|\alpha - p| + 1 = |\beta - p| + 1$. If $p \le \alpha$ or $\beta \le p$, then $\alpha = \beta$, which is impossible. If $\alpha < p < \beta$, then $s = 2p - 2$, contradicting the fact that s is odd.

Subcase 2.2.2.2. x and y are first end-vertices.

Let $x = v_\alpha$ and $y = v_\beta$, where $1 \le \alpha < \beta \le s$. Then by Proposition 3.4.4, $1 \le \alpha \le \left\lceil \frac{s}{2} \right\rceil$ and $\beta = s - \alpha + 2$. Thus, $mr(v_\beta \mid W) = \{s - \beta + 2, |\beta - p| + 2, \beta\}$ $= \{\alpha, |\beta - p| + 2, s - \alpha + 2\}$. Since $mr(v_\alpha \mid W) = \{\alpha, |\alpha - p| + 2, s - \alpha + 2\}$, it follows that $|\alpha - p| + 2 = |\beta - p| + 2$. By the same argument as the proof in Subcase 2.2.2.1, we obtain a contradiction.

Subcase 2.2.2.3. x is a first end-vertex and y is a spine-vertex.

Let $x = v_\gamma$ and $y = u_\delta$, where $1 \le \gamma, \delta \le s$. There are two possibilities:

1) $1 \le \gamma < \delta \le s$.

Then by Proposition 3.4.5 (i), we obtain that $1 \le \gamma \le \left\lceil \frac{s}{2} \right\rceil$ and $\delta = s - \gamma + 3$. Thus, $mr(u_\delta \mid W) = \{s - \delta + 1, |\delta - p| + 1, \delta - 1\} = \{\gamma - 2, |\delta - p| + 1, s - \gamma + 2\}$. Since $mr(v_\gamma \mid W) = \{\gamma, |\gamma - p| + 2, s - \gamma + 2\}$, it follows that $|\gamma - p| + 2 = \gamma - 2$ and $\gamma = |\delta - p| + 1$. Consequently, $|\gamma - p| + 3 = |\delta - p|$. If $p \le \gamma$, then $2\gamma = s$, contradicting the fact that s is odd. If $\gamma < p < \delta$, then $2p = s$, a contradiction. If $\delta \le p$, then $2\gamma - 6 = s$, this is also a contradiction.

2) $1 \le \delta \le \gamma \le s$.

Then by Proposition 3.4.5 (ii), $\left\lceil \frac{s}{2} \right\rceil + 1 \le \gamma \le s$ and $\delta = s - \gamma + 1$. Therefore, $mr(v_\gamma \mid W) = \{s - \gamma + 2, |\gamma - p| + 2, \gamma\} = \{\delta + 1, |\gamma - p| + 2, s - \delta + 1\}$. Since $mr(u_\delta \mid W) = \{\delta - 1, |\delta - p| + 1, s - \delta + 1\}$, it follows that $|\delta - p| + 2 = \delta + 1$ and $\delta - 1 = |\gamma - p| + 2$. Consequently, $|\delta - p| = |\gamma - p| + 3$. If $p < \delta$, then $s = 2\gamma + 2$, contradicting the fact that s is odd. If $\delta \le p \le \gamma$, then $s = 2p - 4$, a contradiction. If $\gamma < p$, then $s = 2\delta + 2$. This is also a contradiction.

Therefore, $\dim_M(T_5) \le 3$ and so $\dim_M(T_5) = 3$, where $p \ge 3$ and s is odd.

Similarly, for $\Psi = \{1, q\}$, where $2 \le q \le s$, $\dim_M(T_5) = 3$ can be proven it the same manner as well.

For an integer $s \ge 4$, let T_6 be a caterpillar $\mathrm{ca}(k_1, k_2, \ldots, k_s)$ such that $\Psi = \{r\}$, where $r \in \{1, 2, \ldots, s\}$. For an integer $s \ge 4$, let T_7 be a caterpillar $\mathrm{ca}(k_1, k_2, \ldots, k_s)$ such that $\Psi = \varnothing$ and $k_r = 1$, where $r \in \{2, 3, \ldots, s-1\}$. Combining Theorem 3.2.1 and Proposition 3.4.8, we arrive yet another result.

Proposition 3.4.9. *A caterpillar T_i, where $6 \le i \le 7$ has multidimension 3.*

Caterpillars with multidimension 3 are completely characterized, as we present next.

Theorem 3.4.10. *For an integer $s \ge 4$, let T be a caterpillar $\mathrm{ca}(k_1, k_2, \ldots, k_s)$. Then T has multidimension 3 if and only if $T = T_i$, where $i \in \{1, 2, \ldots, 7\}$.*

Proof. The preceding results provide the sufficient condition for a caterpillar T having multidimension 3. To show the necessary condition, suppose that T has multidimension 3. By Theorem I, it implies that $|\Psi| \le 3$. For $|\Psi| = 0$, there is an integer r with $2 \le r \le s-1$ such that $k_r = 1$, for otherwise, T is a path, contradicting the fact that $\dim_M(T) = 3$. Hence, $T = T_7$. For $|\Psi| = 1$, obviously, $T = T_6$. It remains therefore only to consider $|\Psi| = 2$ and $|\Psi| = 3$.

For $|\Psi| = 2$, we claim that Ψ contains at least one of $\{1, s\}$. Suppose, contrary to our claim that Ψ contains neither 1 nor s. Let $\Psi = \{r_1, r_2\}$, where $2 \le r_1 < r_2 \le s-1$. By Theorem I, every multibasis of T contains exactly one vertex of $\{v_{r_1}, w_{r_1}\}$, say w_{r_1}. Since there are $\deg_T u_{r_1} = 4$ distinct components of $T - u_{r_1}$, it follows by Proposition 3.2.3 that there is a vertex of a multibasis W belonging to the component containing the spine-vertex $u_{r_1 - 1}$. Similarly, since there are $\deg_T u_{r_2} = 4$ distinct components of $T - u_{r_2}$, there is a vertex of W belonging to the component containing the spine-vertex $u_{r_2 - 1}$. Therefore, W contains at least four vertices, this is a contradiction. Thus, Ψ contains at least one of $\{1, s\}$, that is, $T = T_5$.

For $|\Psi| = 3$, we show that Ψ contains both 1 and s. Assume, to the contrary, that Ψ does not contain 1 or s, say 1. Let $\Psi = \{r_1, r_2, r_3\}$, where $2 \le r_1 < r_2 < r_3 \le s$. Then $W = \{w_{r_1}, w_{r_2}, w_{r_3}\}$ is a multibasis of T. Notice that $\deg_T u_{r_1} = 4$, that is, there are four distinct components of $T - u_{r_1}$. However, both w_{r_2} and w_{r_3} must belong to the same component containing the spine-vertex u_{r_1+1}, contradicting Proposition 3.2.3 that w_{r_1}, w_{r_2} and w_{r_3} cannot belong to the same component of $T - u_{r_1}$. Thus, Ψ contains 1 and s. We may assume without loss of generality that $\Psi = \{1, r, s\}$ with $2 \le r \le \left\lceil \frac{s}{2} \right\rceil$. Then $W = \{w_1, w_r, w_s\}$ is a multibasis of T. If s is even, then $T = T_1$. We may assume that s is odd. If $r = \left\lceil \frac{s}{2} \right\rceil$, then $mr(w_1 \mid W) = mr(w_s \mid W)$, which is impossible. Thus, $2 \le r \le \frac{s-1}{2}$. Next, we consider two cases according to whether s is congruent to 1 or 3 modulo 4.

Case 1. $s \equiv 1 \pmod 4$.

If $r \ne 3$, then $T = T_2$. For $r = 3$, since $r \le \frac{s-1}{2}$, it follows that $s \ge 9$. Next, we claim that $k_{\frac{s-1}{2}} = 0$. Suppose, contrary to our claim, that $k_{\frac{s-1}{2}} \ge 1$. Therefore, $mr(v_{\frac{s-1}{2}} \mid W) = \{\frac{s-3}{2}, \frac{s+1}{2}, \frac{s+5}{2}\} = mr(u_{\frac{s+5}{2}} \mid W)$, contradicting the fact that W is a multibasis of T. Hence, $k_{\frac{s-1}{2}} = 0$ and so $T = T_3$.

Case 2. $s \equiv 3 \pmod 4$.

If $r \ne 3, \frac{s-1}{2}$, then $T = T_2$. For $r = 3$, we claim that $k_{\frac{s-1}{2}} = 0$. Suppose, contrary to our claim, that $k_{\frac{s-1}{2}} \ge 1$. Then $mr(v_{\frac{s-1}{2}} \mid W) = \{\frac{s-3}{2}, \frac{s+1}{2}, \frac{s+5}{2}\} = mr(u_{\frac{s+5}{2}} \mid W)$, contradicting the fact that W is a multibasis of T, as we claimed. Hence, $k_{\frac{s-1}{2}} = 0$ and so $T = T_3$. For $r = \frac{s-1}{2} \ge 4$, since $r \le \frac{s-1}{2}$, it follows that $s \ge 11$. Next, we claim that $k_{\frac{s+5}{4}} = 0$. Suppose, contrary to our claim, that $k_{\frac{s+5}{4}} \ge 1$. Therefore, $mr(v_{\frac{s+5}{4}} \mid W) = \{\frac{s+1}{4}, \frac{s+9}{4}, \frac{3s+3}{4}\} = mr(u_{\frac{3s+3}{4}} \mid W)$, contradicting the fact that that W is a multibasis of T. Hence, $k_{\frac{s+5}{4}} = 0$ and so $T = T_4$.

3.5 The multidimension of symmetric caterpillars

The caterpillars having multidimension 3 are studied in section 3.4. This suggests a way of investigating caterpillars having the multidimension at least 3. Notice that the multidimension of a caterpillar is established by its second end-set Ψ. For a caterpillar $\mathrm{ca}(k_1, k_2, \ldots, k_s)$, observe that for each $1 \le i \le s$, if both i and $s-i+1$ belong to Ψ, then the multirepresentations of second end-vertices w_i and w_{s-i+1} with respect to the set $W = \{w_i \mid i \in \Psi\}$, are the same. This lead us to determine a multibasis of a particular caterpillar. In order to do this, we need an additional definition. For $s \ge 1$, a caterpillar $\mathrm{ca}(k_1, k_2, \ldots, k_s)$ is called a *symmetric caterpillar* if $k_i = k_{s-i+1}$ for each integer i with $1 \le i \le s$. For instance, the symmetric caterpillar $\mathrm{ca}(2,0,2,1,2,0,2)$ is shown in Figure 18.

$\mathrm{ca}(2,0,2,1,2,0,2)$:

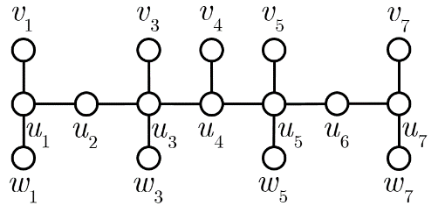

Figure 18: The symmetric caterpillar $\mathrm{ca}(2,0,2,1,2,0,2)$

For $s = 1$, the symmetric caterpillar $\mathrm{ca}(2)$ is a path of order 3. Theorem J implies that its multidimension is 1 with multibasis $\{w_1\}$. For $s = 2$, there are two symmetric caterpillars $\mathrm{ca}(1,1)$ and $\mathrm{ca}(2,2)$. Indeed, $\mathrm{ca}(1,1)$ is a path of order 4 whose multidimension is 1 with multibasis $\{v_1\}$. It is routine to verify that the multidimension of $\mathrm{ca}(2,2)$ is 3 with a multibasis $\{u_1, w_1, w_2\}$. Multibases of these caterpillars are indicated in Figure 19 by solid vertices.

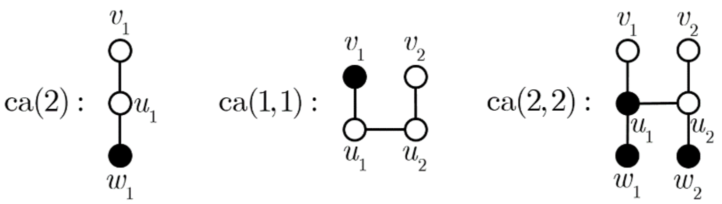

Figure 19: The symmetric caterpillar $\mathrm{ca}(2)$, $\mathrm{ca}(1,1)$ and $\mathrm{ca}(2,2)$

As mentioned earlier, the multidimension of a path is 1. We may therefore consider a symmetric caterpillar that is not a path. For $s \geq 3$, let T be a symmetric caterpillar $\mathrm{ca}(k_1, k_2, \ldots, k_s)$ that is not a path. If $|\Psi| = 0$, then T is a symmetric caterpillar $\mathrm{ca}(k_1, k_2, \ldots, k_s)$ with $k_r = 1$ for some $r \in \{2, 3, \ldots, s-1\}$. Therefore a set $\{u_1, v_1, v_s\}$ is a multibasis of T by Proposition 3.4.9 and so $\dim_M(T) = 3$. If $|\Psi| = 1$, then there is only one integer i belonging to Ψ for some $i \in \{1, 2, \ldots, s\}$. Since T is a symmetric caterpillar, it follows that $i = s - i + 1$, that is, $i = \dfrac{s+1}{2}$. This implies that s is odd and $\Psi = \left\{\dfrac{s+1}{2}\right\}$. Thus, there are two possibilities: (i) $s = 3$ or (ii) $s > 3$. For $s = 3$, the symmetric caterpillar $T = \mathrm{ca}(1, 2, 1)$ has multidimension 4 with a multibasis $\{u_1, v_1, v_3, w_2\}$. For $s > 3$, a symmetric caterpillar T has multidimension 3 with a multibasis $\{u_1, v_s, w_{\frac{s+1}{2}}\}$. We therefore investigate a multibasis and the multidimension of a symmetric caterpillar $\mathrm{ca}(k_1, k_2, \ldots, k_s)$ with $|\Psi| \geq 2$, where $s \geq 3$.

By Observation 3.4.1, the multidimension of a symmetric caterpillar $\mathrm{ca}(k_1, k_2, \ldots, k_s)$ with the second end-set Ψ, must be at least $|\Psi|$. In fact, its multidimension is at least $|\Psi| + 1$, as we now show.

Proposition 3.5.1. *For $s \geq 3$, let T be a symmetric caterpillar $\mathrm{ca}(k_1, k_2, \ldots, k_s)$ with the second end-set Ψ. Then $\dim_M(T) \geq |\Psi| + 1$.*

Proof. If $|\Psi| = 0,$ then the result holds. We may assume for $|\Psi| \geq 1$ that the statement of the proposition is false. Then there is a symmetric caterpillar $\mathrm{ca}(k_1, k_2, \ldots, k_s)$ having a multiresolving set W with $|W| \leq |\Psi|$. By Observation 3.4.1, $|W| = |\Psi|$. However, then, $mr(u_1 \mid W) = mr(u_s \mid W)$, contradicting W as being a multiresolving set of $\mathrm{ca}(k_1, k_2, \ldots, k_s)$.

Proposition 3.5.1 states that $|\Psi| + 1$ is a lower bound for the multidimension of a symmetric caterpillar. Furthermore, we also establish an upper bound for the multidimension of a symmetric caterpillar, as follows.

Proposition 3.5.2. *For $s \geq 3$, let T be a symmetric caterpillar $\mathrm{ca}(k_1, k_2, \ldots, k_s)$ with the second end-set Ψ. Then $\dim_M(T) \leq |\Psi| + 3$.*

Proof. To show $\dim_M(T) \le |\Psi| + 3$, it suffices to verify that there is a multiresolving set of T having cardinality at most $|\Psi| + 3$. Let W be the set of all second end-vertices of T with $|W| = |\Psi|$. We consider three cases for Ψ.

Case 1. 1 belongs to Ψ.

We claim that $B = W \cup \{u_1\}$ is a multiresolving set of T. Assume, contrary to our claim, that there are two vertices x and y of T such that $mr(x \mid W) = mr(y \mid W)$. We consider two subcases.

Subcase 1.1. Both x and y belong to B.

First, we show that both x and y belong to W. Suppose, to the contrary, that either x or y does not belong to W, say x. Then $x = u_1$. Since u_1 and w_1 are the only two adjacent vertices of B, it follows that $mr(u_1 \mid B)$ contains 1 and so $y = w_1$. However, since $d(u_1, w_s)$ and $d(w_1, w_s)$ are the maximum elements of $mr(u_1 \mid B)$ and $mr(w_1 \mid B)$, respectively, it follows that $d(u_1, w_s) = d(w_1, w_s)$, which is a contradiction. Therefore, both x and y must belong to W. Next, we let $x = w_\alpha$ and $y = w_\beta$, where $1 \le \alpha < \beta \le s$. If $1 \le \alpha < \beta \le \left\lceil \frac{s}{2} \right\rceil$, then $d(w_\alpha, w_s) = s - \alpha + 2$ and $d(w_\beta, w_s) = s - \beta + 2$ are the maximum elements of $mr(w_\alpha \mid B)$ and $mr(w_\beta \mid B)$, respectively. Therefore, $\alpha = \beta$, producing a contradiction. Similarly, if $\left\lceil \frac{s}{2} \right\rceil + 1 \le \alpha < \beta \le s$, then $d(w_\alpha, w_1) = \alpha + 1$ and $d(w_\beta, w_1) = \beta + 1$ must be equal, that is, $\alpha = \beta$, which is impossible. We may assume that $1 \le \alpha \le \left\lceil \frac{s}{2} \right\rceil$ and $\left\lceil \frac{s}{2} \right\rceil + 1 \le \beta \le s$. Since $d(w_\alpha, w_s) = s - \alpha + 2$ and $d(w_\beta, w_1) = \beta + 1$ are the maximum elements of $mr(w_\alpha \mid B)$ and $mr(w_\beta \mid B)$, respectively, it follows that $\beta = s - \alpha + 1$. Since T is a symmetric caterpillar, it follows that $mr(w_\alpha \mid W) = mr(w_\beta \mid W)$. However, since $d(w_\alpha, u_1) < d(w_\beta, u_1)$, it follows that $mr(w_\alpha \mid B) \ne mr(w_\beta \mid B)$, this contradicts our assumption.

Subcase 1.2. Neither x nor y belongs to B.

We consider three subcases.

Subcase 1.2.1. x and y are spine-vertices.

Let $x = u_\alpha$ and $y = u_\beta$, where $2 \le \alpha < \beta \le s$. Applying Proposition 3.4.2, we obtain that $2 \le \alpha \le \left\lceil \frac{s}{2} \right\rceil$ and $\beta = s - \alpha + 1$. Since $mr(u_\alpha \mid W) = mr(u_{s-\alpha+1} \mid W)$ and $d(u_\alpha, u_1) < d(u_{s-\alpha+1}, u_1)$, it follows that $mr(u_\alpha \mid B) \ne mr(u_{s-\alpha+1} \mid B)$, which is a contradiction.

Subcase 1.2.2. x and y are first end-vertices.

Let $x = v_\alpha$ and $y = v_\beta$, where $1 \le \alpha < \beta \le s$. By Proposition 3.4.2, $2 \le \alpha \le \left\lceil \frac{s}{2} \right\rceil$ and $\beta = s - \alpha + 1$. Since $mr(v_\alpha \mid W) = mr(v_{s-\alpha+1} \mid W)$ and $d(v_\alpha, u_1) < d(v_{s-\alpha+1}, u_1)$, it follows that $mr(v_\alpha \mid B) \ne mr(v_{s-\alpha+1} \mid B)$, this is also a contradiction.

Subcase 1.2.3. x is a first end-vertex and y is a spine-vertex.

If $\Psi = \{1, s\}$, then it is shown in the proof of Proposition 3.4.3 for $p = 1$ that the set $\{u_1, w_1, w_s\}$ is a multiresolving set of T. We therefore consider the second end-set of cardinality at least 3. Let $p = \min(\Psi - \{1, s\})$. By the symmetry of T, $s - p + 1 = \max(\Psi - \{1, s\})$. Let $x = v_\gamma$ and $y = u_\delta$, where $1 \le \gamma, \delta \le s$. We consider two subcases for γ and δ.

Subcase 1.2.3.1. $1 \le \gamma < \delta \le s$.

By Proposition 3.4.3 (i), $1 \le \gamma \le \left\lceil \frac{s}{2} \right\rceil$ and $\delta = s - \gamma + 2$. Since $d(v_\gamma, w_s)$ and $d(u_\delta, w_1)$ are the maximum elements of $mr(v_\gamma \mid B)$ and $mr(u_\delta \mid B)$, respectively, it follows that $\max(mr(v_\gamma \mid B - \{w_s\})) = d(v_\gamma, w_{s-p+1}) = s - p - \gamma + 3$ and $\max(mr(u_\delta \mid B - \{w_1\})) = d(u_\delta, u_1) = \delta - 1$ are the same. Consequently, $p = 2$ and so $\delta - 1 = s - \gamma + 1$. Since $d(u_\delta, u_1)$ and $d(u_\delta, w_2)$ are in $mr(u_\delta \mid B)$, it follows that $mr(v_\gamma \mid B)$ also contains two $(s - \gamma + 1)$'s. Notice that u_s, v_{s-1} and w_{s-1} are the only three vertices of T whose distance from v_γ is $s - \gamma + 1 = \delta - 1$. Since u_s and v_{s-1} do not belong to B, it follows that $mr(v_\gamma \mid B)$ contains only one element of $s - \gamma + 1$, which contradicts our assumption.

Subcase 1.2.3.2. $1 \le \delta \le \gamma \le s$.

Applying Proposition 3.4.3 (ii), we obtain that $\left\lceil \frac{s}{2} \right\rceil + 1 \le \gamma \le s$ and $\delta = s - \gamma$. Since $d(v_\gamma, w_1)$ and $d(u_\delta, w_s)$ are the maximum elements of $mr(v_\gamma \mid B)$ and

$mr(u_\delta \mid B)$, respectively, it follows that $\max(mr(v_\gamma \mid B-\{w_1\})) = d(v_\gamma, u_1) = \gamma$ and $\max(mr(u_\delta \mid B-\{w_s\})) = d(u_\delta, w_{s-p+1}) = s-p-\delta+2$ are the same. Certainly, $p=2$ and so $\gamma = s-\delta$. Since $d(v_\gamma, u_1)$ and $d(v_\gamma, w_2)$ are in $mr(v_\gamma \mid B)$, it follows that $mr(u_\delta \mid B)$ also contains two $(s-\delta)$'s. Notice that u_s, v_{s-1} and w_{s-1} are the only three vertices of T whose distance from u_δ is $s-\delta=\gamma$. Since u_s and v_{s-1} do not belong to B, it follows that $mr(u_\delta \mid B)$ contains only one element of $s-\delta$, which contradicts our assumption.

Hence, in subcases 1.1 and 1.2 above, $mr(x \mid B) \neq mr(y \mid B)$ for all $x, y \in V(T)$. This implies that B is a multiresolving set of T.

Case 2. 1 and 2 do not belong to Ψ.

Symmetrically, $s-1$ and s also do not belong to Ψ. Let $p = \min(\Psi)$. Then $s-p+1 = \max(\Psi)$. We claim that $B = W \cup \{u_1, v_s\}$ is a multiresolving set of T. Suppose, contrary to our claim, that there are two vertices x and y such that $mr(x \mid B) = mr(y \mid B)$. We consider two subcases.

Subcase 2.1. Both x and y belong to B.

We first show that $u_1, v_s \notin \{x, y\}$. Assume, to the contrary, that $u_1 = x$. Since $\max(mr(u_1 \mid B)) = d(u_1, v_s) = \max(mr(y \mid B))$, it follows that $y = v_s$. However, since $d(u_1, u_1)$ and $d(v_s, v_s)$ are the minimum elements of $mr(u_1 \mid B)$ and $mr(v_s \mid B)$, respectively, and clearly, $\min(mr(u_1 \mid B-\{u_1\})) = d(u_1, w_p) = p$ and $\min(mr(u_1 \mid B-\{u_1\})) = d(v_s, w_{s-p+1}) = p-1$, it follows by Proposition 3.3.3 that $mr(u_1 \mid B) \neq mr(v_s \mid B)$, producing a contradiction. Thus, x and y belong to W. Next, we let $x = w_\alpha$ and $y = w_\beta$, where $p \le \alpha < \beta \le s-p+1$. If $1 \le \alpha < \beta \le \left\lceil \frac{s}{2} \right\rceil$, then $d(w_\alpha, v_s)$ and $d(w_\beta, v_s)$ are the maximum elements of $mr(w_\alpha \mid B)$ and $mr(w_\beta \mid B)$, respectively. Therefore, $\alpha = \beta$, which is a contradiction. Similarly, if $\left\lceil \frac{s}{2} \right\rceil + 1 \le \alpha < \beta \le s$, then $d(w_\alpha, u_1)$ and $d(w_\beta, u_1)$ must be equal, which is impossible. We may assume that $1 \le \alpha \le \left\lceil \frac{s}{2} \right\rceil$ and $\left\lceil \frac{s}{2} \right\rceil + 1 \le \beta \le s$. Since $d(w_\alpha, v_s)$ and $d(w_\beta, u_1)$ are the maximum elements of $mr(w_\alpha \mid B)$ and $mr(w_\beta \mid B)$, respectively, it follows that $\beta = s - \alpha + 2$. Since $\max(mr(w_\alpha \mid B - \{v_s\})) = d(w_\alpha, w_{s-p+1}) = s - p + \alpha + 3$ and $\max(mr(w_\beta \mid B - \{u_1\})) = d(w_\beta, w_p) = \beta - p + 2$ are equal, it follows that, certainly, $\beta = s - \alpha + 1$, producing a contradiction.

Subcase 2.2. Neither x nor y belongs to B.

We consider three subcases.

Subcase 2.2.1. x and y are spine-vertices.

Let $x = u_\alpha$ and $y = u_\beta$, where $2 \le \alpha < \beta \le s$. By Applying Proposition 3.4.4, it implies that $2 \le \alpha \le \left\lceil \frac{s}{2} \right\rceil$ and $\beta = s - \alpha + 2$. Since $d(u_\alpha, v_s)$ and $d(u_\beta, u_1)$ are the maximum elements of $mr(u_\alpha \mid B)$ and $mr(u_\beta \mid B)$, respectively, $\max(mr(u_\alpha \mid B - \{v_s\})) = d(u_\alpha, w_{s-p+1}) = s - p - \alpha + 2$ and $\max(mr(u_\beta \mid B - \{u_1\})) = d(u_\beta, w_p) = \beta - p + 1$ must be equal. Necessarily, then $\beta = s - \alpha + 1$. This is a contradiction.

Subcase 2.2.2. x and y are first end-vertices.

Let $x = v_\alpha$ and $y = v_\beta$, where $1 \le \alpha < \beta \le s$. By Proposition 3.4.4, $1 \le \alpha \le \left\lceil \frac{s}{2} \right\rceil$ and $\beta = s - \alpha + 2$. Since $d(v_\alpha, v_s)$ and $d(v_\beta, u_1)$ are the maximum elements of $mr(v_\alpha \mid B)$ and $mr(v_\beta \mid B)$, respectively, it follows that $\max(mr(v_\alpha \mid B - \{v_s\})) = d(v_\alpha, w_{s-p+1}) = s - p - \alpha + 3$ and $\max(mr(v_\beta \mid B - \{u_1\}))$ $= d(v_\beta, w_p) = \beta - p + 2$ must be equal. Consequently, $\beta = s - \alpha + 1$. This is also a contradiction.

Subcase 2.2.3. x is a first end-vertex and y is a spine-vertex.

Let $x = v_\gamma$ and $y = u_\delta$, where $1 \le \gamma, \delta \le s$. We consider two subcases according to γ and δ.

Subcase 2.2.3.1. $1 \le \gamma < \delta \le s$.

By Proposition 3.4.5 (i), $1 \le \gamma \le \left\lceil \frac{s}{2} \right\rceil$ and $\delta = s - \gamma + 3$. Since $d(v_\gamma, v_s)$ and $d(u_\delta, u_1)$ are the maximum elements of $mr(v_\gamma \mid B)$ and $mr(u_\delta \mid B)$, respectively, it follows that $\max(mr(v_\gamma \mid B - \{v_s\})) = d(v_\gamma, w_{s-p+1}) = s - p - \gamma + 3$ and

$\max(mr(u_\delta \mid B-\{u_1\})) = d(u_\delta, w_p) = p-\delta+1$ must be equal. Evidently, $\delta = s-\gamma+2$, which is impossible.

Subcase 2.2.3.2. $1 \le \delta \le \gamma \le s.$

Applying Proposition 3.4.5 (ii), it implies that $\left\lceil \frac{s}{2} \right\rceil + 1 \le \gamma \le s$ and $\delta = s-\gamma+1,$ Since $d(v_\gamma, u_1)$ and $d(u_\delta, v_s)$ are the maximum elements of $mr(v_\gamma \mid B)$ and $mr(u_\delta \mid B)$, respectively, it follows that $\max(mr(v_\gamma \mid B-\{u_1\})) = d(v_\gamma, w_p) = \gamma - p + 2$ and $\max(mr(u_\delta \mid B-\{v_s\})) = d(u_\delta, w_{s-p+1}) = s-p-\delta+2$ are equal. As verified above, $\delta = s-\gamma$, which cannot occur.

Hence, $mr(x \mid B) \ne mr(y \mid B)$ for all $x, y \in V(T)$. This implies that B is a multiresolving set of T.

Case 3. 1 does not belong to Ψ and 2 belongs to Ψ.

Let T' be a symmetric caterpillar which is obtained from T by joining end-vertices x and y to the spine-vertices u_1 and u_s, respectively. Therefore, $1 \in \Psi_{T'}$. By applying Case 1, $B = (W \cup \{v_1, v_s\}) \cup \{u_1\}$ is a multiresolving set of T'. Since $T = T' - \{x, y\}$, it follows by Corollary 3.2.2 that $B = W \cup \{u_1, v_1, v_s\}$ is a multiresolving set of T.

Hence, every symmetric caterpillar T has a multiresolving set of cardinality at most $|\Psi| + 3$ and so $\dim_M(T) \le |\Psi| + 3$.

The following result is obtained from the bounds given in Propositions 3.5.1 and 3.5.2.

Corollary 3.5.3. *For $s \ge 3$, let T be a symmetric caterpillar $\mathrm{ca}(k_1, k_2, \dots, k_s)$ with the second end-set Ψ. Then*

$$|\Psi| + 1 \le \dim_M(T) \le |\Psi| + 3.$$

The multibases of symmetric caterpillars are characterized by the following result. Furthermore, the sharpness of Corollary 3.5.3. is presented.

Theorem 3.5.4. *For $s \ge 3$, let T be a symmetric caterpillar $\mathrm{ca}(k_1, k_2, \dots, k_s)$ with $|\Psi| \ge 2$ and let W be a set of all second end-vertices of T. Then*

(i) *if $1 \in \Psi$, then $W \cup \{u_1\}$ is a multibasis of T,*

(ii) *if* $1,2 \notin \Psi$ *, then* $W \cup \{u_1, v_s\}$ *is a multibasis of* T*, and*

(iii) *if* $1 \notin \Psi$ *and* $2 \in \Psi$ *, then* $W \cup \{u_1, v_1, v_s\}$ *is a multibasis of* T*.*

Proof. (i) Assume that $1 \in \Psi$. By Case 1 in the proof of Proposition 3.5.2, it implies that $W \cup \{u_1\}$ is a multiresolving set of T. Hence, $\dim_M(T) = |\Psi| + 1$ by Corollary 3.5.3, and so $W \cup \{u_1\}$ is a multibasis of T.

(ii) Assume that $1,2 \notin \Psi$. By Case 2 in the proof of Proposition 3.5.2, it implies that $W \cup \{u_1, v_s\}$ is a multiresolving set of T. Therefore, $\dim_M(T) \leq |\Psi| + 2$. Next, we claim that $\dim_M(T) \geq |\Psi| + 2$. Let $p = \min(\Psi)$. Since there are four components of $T - u_p$, it follows by Theorem I and Proposition 3.2.3 that at least one vertex from the component of $T - u_p$ containing a vertex u_{p-1}, belongs to every multiresolving set of T. Similarly, every multiresolving set must contain at least one vertex from the component of $T - u_{s-p+1}$ containing u_{s-p+2}. Therefore, by Observation 3.4.1, every multiresolving set of T has cardinality at least $|\Psi| + 2$. Hence, $\dim_M(T) = |\Psi| + 2$ and so $W \cup \{u_1, v_s\}$ is a multibasis of T.

(iii) Assume that $1 \notin \Psi$ and $2 \in \Psi$. By Case 3 in the proof of Proposition 3.5.2, it implies that $W \cup \{u_1, v_1, v_s\}$ is a multiresolving set of T. Thus, $\dim_M(T) \leq |\Psi| + 3$. Next, we show that $\dim_M(T) \geq |\Psi| + 3$. Since there are four components of $T - u_2$, it follows by Proposition 3.2.3 that every multiresolving set of T must contain at least one vertex of $\{u_1, v_1\}$. Similarly, every multiresolving set of T must contain at least one vertex of $\{u_s, v_s\}$. We claim that every multiresolving set of T contains three vertices of $\{u_1, u_s, v_1, v_s\}$. Suppose, contrary to our claim, that there is a multiresolving set S of T containing only one of $\{u_1, v_1\}$ and one of $\{u_s, v_s\}$. By Observation 3.4.1, we may assume without loss of generality, that $W \subset S$. If $u_1, u_s \in S$, then $mr(u_1 \mid S) = mr(w_2 \mid S)$, which is impossible. If $u_1, v_s \in S$, then $mr(u_1 \mid S) = mr(w_2 \mid S)$, a contradiction. If $v_1, u_s \in S$, then $mr(u_s \mid S) = mr(w_{s-1} \mid S)$, producing a contradiction. If $v_1, v_s \in S$, then $mr(v_1 \mid S) = mr(v_s \mid S)$, which is also impossible. Therefore, every multiresolving set of T has cardinality at least $|\Psi| + 3$. Hence, $\dim_M(T) = |\Psi| + 3$ and so $W \cup \{u_1, v_1, v_s\}$ is a multibasis of T.

Let T be a symmetric caterpillar $\mathrm{ca}(k_1, k_2, \ldots, k_s)$ with $k_i = 1$ for some integer i with $2 \le i \le s-1$. If $T - v_i$ is not a path, then by applying Theorems 3.2.1 and 3.5.4, $\dim_M(T - v_i) = \dim_M(T)$. This observation provides us with the following more general result as we state next.

Corollary 3.5.5. *For $s \ge 3$, let T be a symmetric caterpillar $\mathrm{ca}(k_1, k_2, \ldots, k_s)$ with the second end-set Ψ_T and let T' be a caterpillar $\mathrm{ca}(l_1, l_2, \ldots, l_s)$ that is a subgraph of T and is not a path, with the second end-set $\Psi_{T'}$. If $\Psi_{T'} = \Psi_T$, then $\dim_M(T') = \dim_M(T)$.*

Proof. Suppose that $\Psi_{T'} = \Psi_T$. If $T' = T$, then $\dim_M(T') = \dim_M(T)$. We therefore assume that $T' \ne T$. Since T' is a subgraph of T, it follows that there is an integer i with $2 \le i \le s-1$ such that $k_i = 1$ but $l_i = 0$. Symmetrically, $k_{s-i+1} = 1$ and $l_{s-i+1} = 0$. Let $F = \{v_i \in V(T) \mid l_i = 0\}$. Note that $T' = T - F$. Theorem 3.5.4 implies that every multibasis of T does not contain every first end-vertex in F. Therefore, a multibasis of T is also a multibasis of T' and so $\dim_M(T') = \dim_M(T)$.

CHAPTER 4
CONCLUSION AND OPEN PROBLEMS

We conclude main results of this work and give some open problems for future work in this chapter.

4.1 Conclusion

This section is to present our comprehensive work concerning the connected local dimension and the multidimension of graphs. The main results are as follows:

4.1.1 The connected local dimension of graphs

4.1.1.1 The connected local dimensions of some well-known graphs.

1. Let G be a connected graph of order $n \geq 2$. Then
   (i) $\mathrm{cld}(G) = 1$ if and only if G is a bipartite graph,
   (ii) $\mathrm{cld}(G) = n-1$ if and only if $G = K_n$, a complete graph of order n.
2. For an integer $n \geq 3$, the connected local dimension of a cycle C_n is
$$\mathrm{cld}(C_n) = \begin{cases} 1 & \text{if } n \text{ is even,} \\ 2 & \text{if } n \text{ is odd.} \end{cases}$$
3. Let W_n be a wheel, where $n \geq 7$. Then $\mathrm{cld}(W_n) = \left\lceil \frac{n}{4} \right\rceil + 1.$

4.1.1.2 Graphs with prescribed connected local dimensions and other parameters

1. Let a, b and n be integers with $n \geq 4$. Then there exists a connected graph G of order n with $\mathrm{ld}(G) = a$ and $\mathrm{cld}(G) = b$ if and only if a, b, n satisfy one of the following:
   (i) $a = b = n-1,$
   (ii) $a = b = 1,$ and
   (iii) $2 \leq a \leq b \leq n-2.$
2. Let b, c and n be integers with $n \geq 4$. Then there exists a connected graph G of order n with $\mathrm{cld}(G) = b$ and $\mathrm{cd}(G) = c$ if and only if b, c, n satisfy one of the following:

(i) $b = c = n-1,$

(ii) $b = 1$ and $1 \leq c \leq n-1,$ and

(iii) $2 \leq b \leq c \leq n-2.$

4.1.1.3 Connected local bases and local bases in graphs

1. There is an infinite class of connected graph G such that some connected local bases of G contain a local basis of G and others contain no local basis of G.
2. For $k \geq 3,$ there exists a graph with a unique connected local basis of cardinality $k+1$.

4.1.2 The multidimension of graphs

4.1.2.1 The multisimilar classes of graphs

1. Let W be a set of vertices of a connected graph G and let u and v be vertices of G such that $u \in [v]_W$. Then $mr(u \mid W)$ and $mr(v \mid W)$ have the same minimum (or maximum) element if and only if $mr(u \mid W) = mr(v \mid W)$.
2. If W is a multiresolving set of a connected graph $G,$ then the cardinality of multisimilar class of each vertex of G with respect to W is at most $\mathrm{diam}(G)+1$.
3. Let u and v be vertices of a connected graph G and let W be a set of vertices of $G.$ Then

   (i) if $[u]_W \neq [v]_W,$ then $mr(x \mid W) \neq mr(y \mid W)$ for all $x \in [u]_W$ and $y \in [v]_W$,

   (ii) if $[u]_W = \{u\}$ for all $u \in V(G),$ then W is a multiresolving set of $G.$

4.1.2.2 The characterization of caterpillars with multidimension 3

1. A caterpillar T_i, where $1 \leq i \leq 4$ has multidimension 3.
2. A caterpillar T_5 has multidimension 3.
3. A caterpillar T_i, where $6 \leq i \leq 7$ has multidimension 3.

4. For an integer $s \geq 4$, let T be a caterpillar $\mathrm{ca}(k_1, k_2, \ldots, k_s)$. Then T has multidimension 3 if and only if $T = T_i$, where $i \in \{1, 2, \ldots, 7\}$.

4.1.2.3 The multidimension of symmetric caterpillars

1. For $s \geq 3$, let T be a symmetric caterpillar $\mathrm{ca}(k_1, k_2, \ldots, k_s)$ with the second end-set Ψ. Then $\dim_M(T) \geq |\Psi| + 1$.
2. For $s \geq 3$, let T be a symmetric caterpillar $\mathrm{ca}(k_1, k_2, \ldots, k_s)$ with the second end-set Ψ. Then $\dim_M(T) \leq |\Psi| + 3$.
3. For $s \geq 3$, let T be a symmetric caterpillar $\mathrm{ca}(k_1, k_2, \ldots, k_s)$ with the second end-set Ψ. Then $|\Psi| + 1 \leq \dim_M(T) \leq |\Psi| + 3$.
4. For $s \geq 3$, let T be a symmetric caterpillar $\mathrm{ca}(k_1, k_2, \ldots, k_s)$ with $|\Psi| \geq 2$ and let W be a set of all second end-vertices of T. Then
   (i) if $1 \in \Psi$, then $W \cup \{u_1\}$ is a multibasis of T,
   (ii) if $1, 2 \notin \Psi$, then $W \cup \{u_1, v_s\}$ is a multibasis of T, and
   (iii) if $1 \notin \Psi$ and $2 \in \Psi$, then $W \cup \{u_1, v_1, v_s\}$ is a multibasis of T.
5. For $s \geq 3$, let T be a symmetric caterpillar $\mathrm{ca}(k_1, k_2, \ldots, k_s)$ with the second end-set Ψ_T and let T' be a caterpillar $\mathrm{ca}(l_1, l_2, \ldots, l_s)$ that is a subgraph of T and is not a path, with the second end-set $\Psi_{T'}$. If $\Psi_{T'} = \Psi_T$, then $\dim_M(T') = \dim_M(T)$.

4.2 Open problems

In Chapter 2, we know by (2.3) that $1 \leq \mathrm{ld}(G) \leq \mathrm{cld}(G) \leq \mathrm{cd}(G) \leq n - 1$. It suggests the following question: For which quadruples a, b, c, n of integers with $1 \leq a \leq b \leq c \leq n - 1$, does there exist a connected graph G of order n with $\mathrm{ld}(G) = a, \mathrm{cld}(G) = b$ and $\mathrm{cd}(G) = c$?

In Chapter 3, the complete graph K_n is only one graph that its dimension is $n - 1$ but not so for multidimensions. It follows by (15) and (16) that the multidimension of complete graph is not defined. Thus, (3.1) leads us to the conjecture: If G is a connected graph such that $\dim_M(G)$ is defined, then $\dim_M(G) \leq n - 2$.

In section 3.4, for an integers $s \geq 2$, let T be a caterpillar $\mathrm{ca}(k_1, k_2, \ldots, k_s)$ of order n such that $\Psi \neq \varnothing$ and $\dim_M(T)$ is defined. It then follows by Theorem I that

$$\left|\Psi\right| \leq \dim_M(T) \leq n - \left|\Psi\right|.$$

Moreover, by Corollary 3.4.7, caterpillars T_1, T_2, T_3 and T_4 also illustrate the sharpness of this lower bound. It would be interesting to determine whether this upper bound is sharp or not.

In section 3.5, a subdivision T' of a symmetric caterpillar T is a graph that is obtained from T by inserting vertices of degree 2 into some, all or none of the edge of T. It would be interesting to study a multibasis of a subdivision T' of T.

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VITA

NAME	Supachoke Isariyapalakul
DATE OF BIRTH	5 December 1987
PLACE OF BIRTH	Bangkok
INSTITUTIONS ATTENDED	2014 M.S. (Mathematics), Kasetsart University, Bangkok, Thailand 2009 B.Sc. (Mathematics), Srinakharinwirot University, Bangkok, Thailand
HOME ADDRESS	1626/106 Dindang, Dindang, Bangkok, Thailand 10400