THE NUMERICAL STUDIES OF THE MODIFIED VAN DER POL EQUATION IN ELECTRICAL CIRCUITS

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NATTAPON CHOTSISUPARAT

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THE NUMERICAL STUDIES OF THE MODIFIED VAN DER POL EQUATION
IN ELECTRICAL CIRCUITS

BY

NATTAPON CHOTSISUPARAT

HAS BEEN APPROVED BY THE GRADUATE SCHOOL IN PARTIAL FULFILLMENT
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The van der Pol equation is a nonlinear dynamical system found in a triode circuit. The modified van der Pol equation is defined as $d^2x/dt^2-u(1-x^2)dx/dt+x^3=B\cos(t)$. This research studied the modified van der Pol equation numerically. We plotted the graph for (x,y), (d^2x/dt^2,y) and a time series with different parameters u,B,x(0),y(0) and a number of steps to study their behaviors, using the Runge–Kutta method (RK4), and the Adams-Bashforth-Moulton method (ABM). The results of the phase plane analysis showed that among a total of eight results, two were found to have a limit cycle, and the other six were nonexistence of a limit cycle. It was concluded that in the case of a limit cycle does not depend on the initial condition. A van der Pol equation and the modified van der Pol equation were compared. The case of almost periodic oscillation and power spectral density (PSD) were also investigated.

Keyword : The modified van der Pol equation, non-linear system, second order differential equations, chaotic behavior in systems, almost periodic oscillations

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CHAPTER 1
INTRODUCTION

1 Introduction

In this research, the van der Pol oscillator is studied. We can write down the following equation

$$\frac{d^2x}{dt^2} - \mu(1 - x^2)\frac{dx}{dt} + x = 0. \quad (1.1)$$

This is a second-order differential equation where μ is a parameter that characterizes the nonlinear term. Many systems in nature are nonlinear systems such as Navier–Stokes equations, and the nonlinear Schrödinger equation. The van der pol equation is a famous nonlinear system. There are also other interesting examples which emphasize nonlinear system. This equation was originally proposed by engineer named Balthasar van der Pol in 1926 (Van der Pol, 1926).

After van der Pol proposed his equation, there have been many research articles about the van der Pol oscillator. In 1949 N. Minorsky discussed energy fluctuations (Minorsky, 1949). In 1979 D.A. Linkens used weakly coupled van der Pol oscillators which can be used to analyze a biological system with beating and modulation phenomena (Linkens, 1979). For many years the van der Pol oscillator has been popular among researchers and has continued to be used for research topics.

In this research we focused on the modified van der Pol equation,

$$\frac{d^2x}{dt^2} - \mu(1 - x^2)\frac{dx}{dt} + x^3 = B\cos(t). \quad (1.2)$$

which Ueda and Akamatsu investigated in their 1981 papers (Ueda & Akamatsu, 1981). We will study the chaotic behavior of the modified van der Pol equation and study the case of almost periodic oscillation in the forced negative-resistance oscillator. We used Fortran 90, an imperative programming language, as a tool for calculation and gnuplot, a command line program for plotting the data.

This research helped us obtain a better understanding of dynamical and chaotic systems that may be applied to many fields in science and engineering.

2 The Purpose of the Research

1. To study the history of the development of the van der Pol equation.

2. To calculate the solutions and study their chaotic behaviors in; phase space, time series, power spectrum and limit cycles of the modified van der Pol equation.

3. To compare the properties and characteristics between the van der Pol equation and the modified van der Pol equation.

4. To study the case of almost periodic oscillation in the forced negative-resistance oscillator.

3 The Importance of the Research

To describe the characteristics of the van der Pol equation especially the modified van der Pol equation which is a mathematical model of the systems with self-excited, limit cycle oscillations. This mathematical model can be applied to many systems in science and engineering. This research will give us a better understanding of chaotic behavior and non-linear systems.

4 Scope of the Research

Our primary scope is to use the numerical methods for differential equations by Runge–Kutta methods (RK4), and the Adams-Bashforth-Moulton methods (ABM) to calculate behaviors in phase space, time series and limit cycles. For the power spectra we use a fast Fourier transform (FFT). This research aims to study two main cases; the van der Pol equation and the modified van der Pol equation, and to also study the case of almost periodic oscillation in the forced negative-resistance oscillator.

CHAPTER 2
REVIEW OF THE LITERATURE

In this chapter, the theoretical background and literature review will be discussed. First, we will review the introduction and history of the van der Pol oscillator, then the Duffing equation will be discussed briefly. After that we introduce the mathematical theory about the van der Pol oscillator which are averaging theory and the Liénard theorem. We will review papers about non-linearity in the circuit systems, discuss the differential equations and numerical methods that are related and used in this research and finally, the related research and applications will be examined.

1 Introduction of the van der Pol Equation

In 1920 van der Pol published a paper called "A theory of the amplitude of free and forced triode vibrations". He experimented with the oscillations in a vacuum tube triode circuit and found that all initial conditions converged to the same periodic orbit of finite amplitude (Tsatsos, 2006). In 1920 Balthasar van der Pol showed a differential equation which we called an unforced van der Pol oscillation

$$\frac{d^2x}{dt^2} - \mu(1 - x^2)\frac{dx}{dt} + x = 0. \tag{2.1}$$

We also see this equation in the theory of lasers (DeVries & Hasbun, 2011). If $\mu = 0$, equation will be a simple harmonic oscillator and if $\mu \neq 0$, it will be a nonlinear equation (DeVries & Hasbun, 2011).

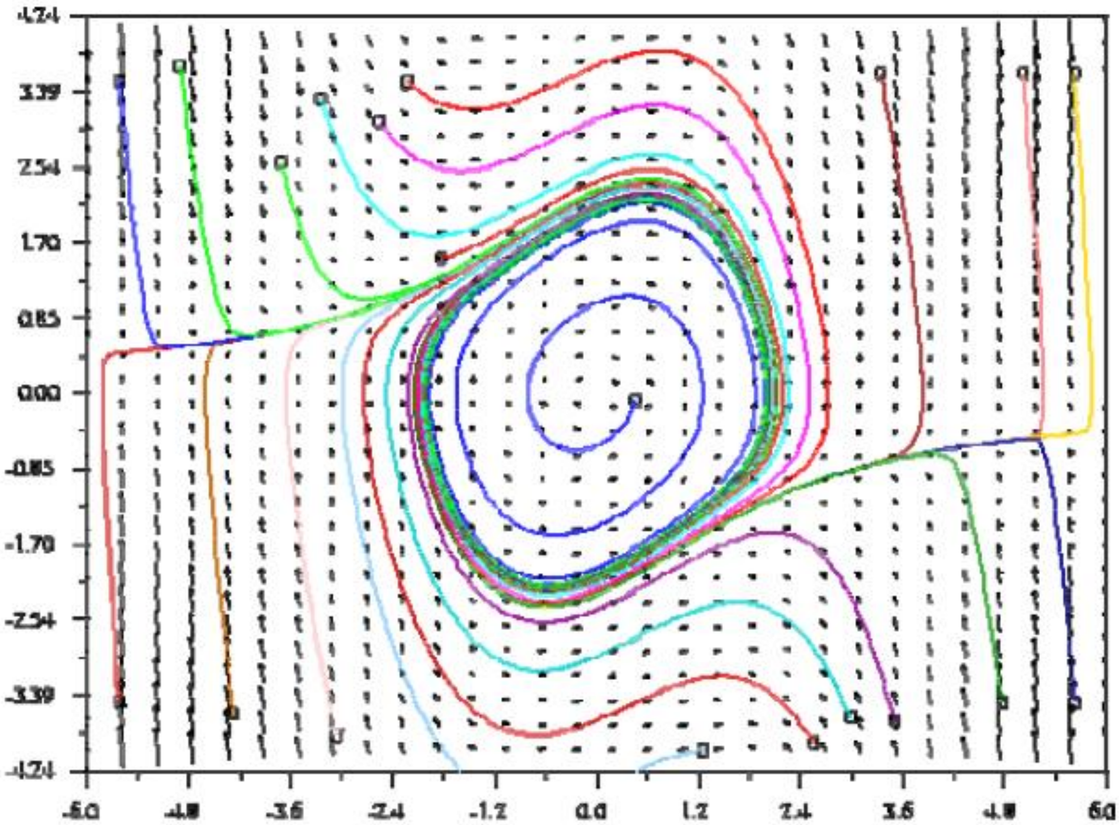

Figure 1 The behavior in a phase space of a self-oscillatory system

Source: Marios Tsatsos. (2006). Theoretical and Numerical Study of the Van der Pol equation p. 8

In 1926 van der Pol published a paper called “Relaxation oscillations” (Van der Pol, 1926). In this paper he showed that in all cases the system had relaxation-oscillations. Relaxation oscillations are a type of limit cycle.

In 1927 he published a paper entitled “Frequency Demultiplication” (Van der Pol & Van Der Mark, 1927) in Nature. He went further and introduced a forced van der Pol equation that included a periodic forcing term,

$$\frac{d^2x}{dt^2} - \mu(1 - x^2)\frac{dx}{dt} + x = a\sin(\omega t). \tag{2.2}$$

He found that there is a strange sound in electronic circuit modelled by (2.2) and he also found the existence of stable periodic solutions with different periods for particular values of the parameters.

In September 1934 van der Pol published a paper entitled “The Nonlinear Theory of Electric Oscillations” (Van der Pol, 1934). It appeared that many periodic phenomena, which explanations of the basis of sinusoidal oscillations were based upon, failed to give a satisfactory explanation. But we can perform better studies from the relaxation oscillation point of view. For a general consideration of relaxation oscillations we can study many phenomena such as the beating of the heart with its many anomalies. These show ab origine frequency demultiplication, aerodynamic phenomena associated with eddies, and periodic reoccurrence of economical crises.

Van der Pol also found that, using (2.2) oscillation in the regime of relaxation oscillation triggered subharmonical oscillations appearance which were due to “irregular noise” before the transition from one subharmonical regime to another. This observation is one of the early researches that showed the chaotic oscillations in the electronic circuit. It is now a branch of mathematical physics concerning the behavior of dynamical systems that are highly sensitive to initial conditions. It is called the chaos theory (Tsatsos, 2006).

The van der Pol oscillator is named after a Dutch physicist Balthasar van der Pol (1889 – 1959). He had studied for his PhD in physics at Utrecht University and later worked at Philips Research Labs. His PhD thesis is entitled “The effect of an ionized gas on electro-magnetic wave propagation and its application to radio”. He was also interested in mathematics, especially differential equations in coupled electrical systems. In 1926 he published a scientific paper entitled “On relaxation-oscillations” in the Philosophical Magazine. This paper shows that the system has relaxation-oscillations which are today called limit cycles.

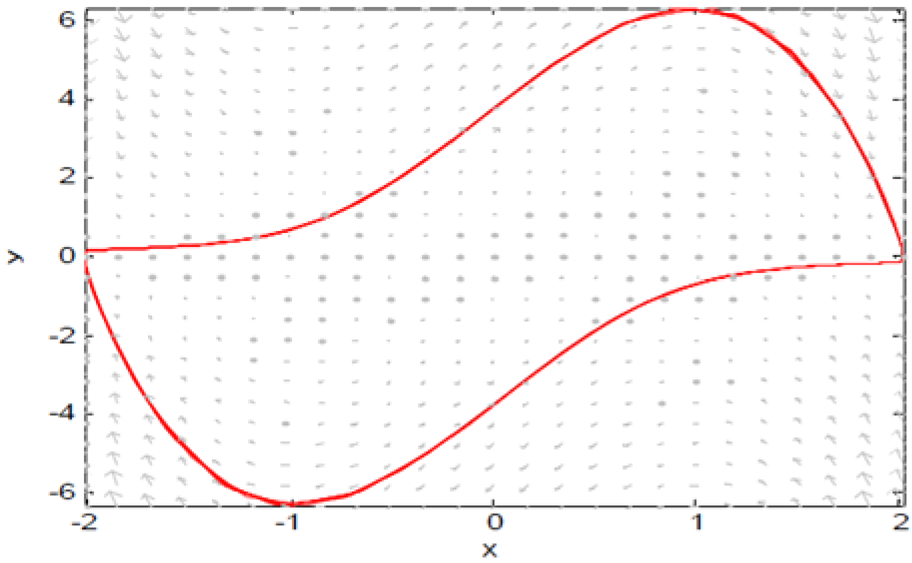

Figure 2 Limit cycle and direction of field for unforced van der Pol oscillator

Source: P. Sivák. (2014). Some Methods of Analysis of Chaos in Mechanical Systems p. 201

At first, we have a linear second-order ordinary differential equation,

$$\frac{d^2x}{dt^2} + \alpha\frac{dx}{dt} + \omega^2 x = 0. \quad (2.3)$$

The solution is

$$x = C_1 e^{\frac{-\alpha t}{2}} \sin(\sqrt{\omega^2 - \left(\frac{\alpha^2}{4}\right)}\, t + \phi), \quad (2.4)$$

where

$$\alpha > 0 \text{ and } \left(\frac{\alpha^2}{4}\right) < \omega^2. \quad (2.5)$$

(2.4) will be a damped oscillation with a logarithmic decrement δ,

$$\frac{\delta}{\pi} = \frac{\alpha}{\omega}. \quad (2.6)$$

In electrical circuit systems the resistance is negative and a sign of α is reversed, and then the equation becomes

$$\frac{d^2x}{dt^2} - \alpha\frac{dx}{dt} + \omega^2 x = 0. \quad (2.7)$$

And the solution becomes

$$x = C_1 e^{\frac{+\alpha t}{2}} \sin(\sqrt{\omega^2 - \left(\frac{\alpha^2}{4}\right)}\, t + \phi), \tag{2.8}$$

where

$$\alpha > 0 \text{ and } \left(\frac{\alpha^2}{4}\right) < \omega^2. \tag{2.9}$$

This is an oscillation but the amplitude is gradually increasing instead of decreasing with a logarithmic decrement δ

$$\frac{\delta}{\pi} = \frac{\alpha}{\omega}. \tag{2.10}$$

The amplitude is growing to infinity which is impossible in the physical world, the value of x must be valid up to a certain value. To limit the value of amplitude we replace α with $\alpha - 3yx^2$ where y is a constant so the equation will become

$$\frac{d^2x}{dt^2} - (\alpha - 3yx^2)\frac{dx}{dt} + \omega^2 x = 0. \tag{2.11}$$

Then we change the unit of time and x,

$$\omega t = t' \text{ and } x = \sqrt{\frac{\alpha}{3y}}\, v. \tag{2.12}$$

The equation becomes

$$\frac{d^2v}{dt^2} - \frac{\alpha}{\omega}\,(1 - v^2)\frac{dv}{dt} + v = 0. \tag{2.13}$$

And if $\mu = \frac{\alpha}{\omega}$ then we will have

$$\frac{d^2v}{dt^2} - \mu(1 - v^2)\frac{dv}{dt} + v = 0. \tag{2.14}$$

For the quasi-aperiodic case which is different from a normal approximately sinusoidal solution, or a purely periodic solution, where the time period is expressed by the time of relaxation of the system. In his paper van der Pol called this phenomenon “relaxation-oscillation” (Van der Pol, 1926).

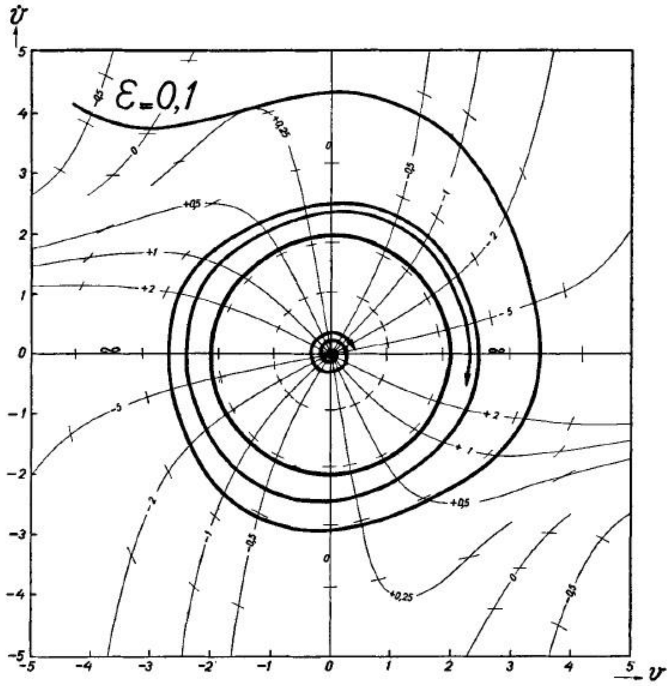

Figure 3 A quasi-aperiodic case, the final steady closed curve representing the periodic solution

Source: B. van der Pol. (1926). On "relaxation-oscillations" p. 983

2 History of the van der Pol Equation

Van der Pol published the concept of relaxation oscillation such as in a triode circuit system in Philosophical Magazine in 1926. But before that there were many people who had done work related to this topic. Starting from 1880, Gérard-Lescuyer discussed the series dynamo machine in relation to the van der Pol equation as described in his 1926 paper.

Gérard-Lescuyer may be the first scientist who reported this phenomenon. He worked on the manufacture of electrical generators where he coupled a dynamo acting as a generator to a magneto-electric machine. He noticed that even when the source of current was constant there were periodic reversals in the rotation of the magneto-electric machine. This phenomenon was later explained that between the brushes of the dynamo there was an electromotive force that could be represented by a nonlinear function of the current.

Back in the 19th century people used the electric arc for night light but there was a problem with the disturbing sound of electrical discharge. William Duddell an English physicist and electrical engineer came up with the idea to solve this problem by inserting a high frequency oscillating circuit into the circuit. Finally, the problem was solved because people cannot hear the sound at frequency approaching 30 000 oscillations per second. Nevertheless, Duddell found something interesting after he built the oscillating electric circuit with an electrical arc and a constant source. He found that when the arc was supplied with an oscillating current, it created a humming frequency which was periodic and a hissing frequency which was irregular. He also found when the arc was supplied with a constant current, it created a self-oscillating electrical arc. He called this phenomenon a musical arc.

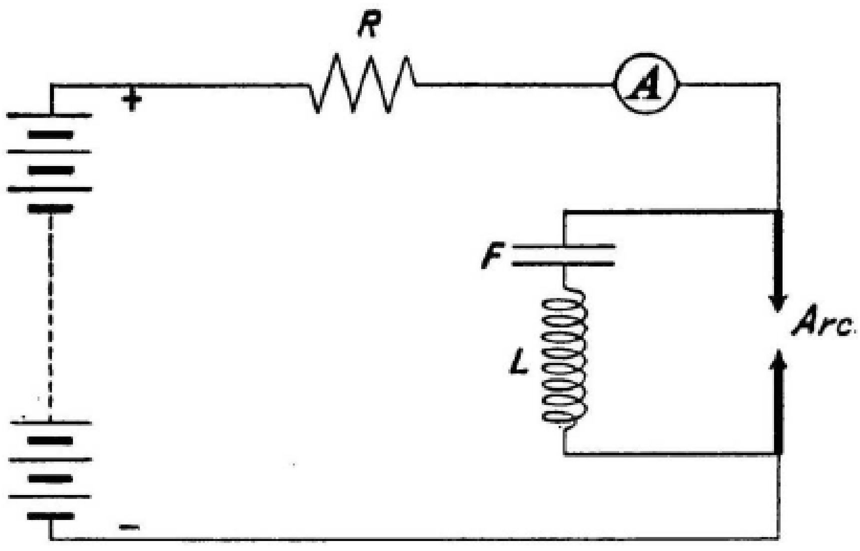

Figure 4 Electric arc with an oscillating circuit by Duddell

Source: W. Duddell. (1904). On rapid variations in the current through the direct-current arc p. 6

Henri Poincaré (1854-1912), a French mathematician developed the concept of limit cycle and the theory of differential equations. He gave a series of lectures which talked about the existence of sustained oscillations in the musical arc that corresponded to a limit cycle.

In 1919 Paul Janet found the similarity of three electronic devices including the series of dynamo machine, the musical arc and the triode (Ginoux & Letellier, 2012). André Blondel developed a triode equation to characterize the circuit with an audion. He proposed this approximate equation

$$C\frac{d^2u}{dt^2} - (b_1h - 3b_3h^3u^2 - \cdots)\frac{du}{dt} + \frac{u}{L} = 0. \qquad (2.15)$$

After Blondel's proposed his equation was just one year old, van der Pol introduced his equation which was slightly different from Blondel's by neglecting a resistance R in the circuit and also by introducing a parallel to the capacitor (Van der Pol, 1920).

And finally, in 1926 van der Pol introduced his famous equation. It is a dimensionless equation

$$\frac{d^2x}{dt^2} - \mu(1 - x^2)\frac{dx}{dt} + x = 0. \qquad (2.16)$$

This equation became famous with scientists and engineers especially in the field of radio and electricity. Because this equation is dimensionless it can be useful to many fields in science and mathematics. After 1926 van der Pol did more research for some solutions for his equation.

In conclusion, the van der Pol equation is the first self-oscillating equation, and also, the first reduced and dimensionless equation. His works on relaxation oscillation and non-linear phenomenon is important for upcoming scientists and engineers.

3 Duffing Equation

The van der Pol equation is a model of an electronic circuit found in very early radios in the 1920s (Chen & Chen, 2008). The van der Pol equation describes self-sustaining oscillations in which energy is fed into small oscillations and is removed from large oscillations. Another system that is worth a mention is the Duffing equation, a nonlinear second order differential equation discovered by Georg Duffing. The van der Pol and Duffing equations are the most common examples of nonlinear oscillation in textbooks and research articles. It is interesting to study and compare these two equations.

The Duffing equation is

$$\frac{d^2x}{dt^2} + \delta\frac{dx}{dt} + \alpha x + \beta x^3 = \gamma\cos(\omega t). \tag{2.17}$$

Where α is the stiffness, β is the amount of non-linearity of the restoring force, γ is the amplitude of the driving force, δ is the amount of damping and ω is the angular frequency.

In general, the Duffing equation does not provide an exact analytic solution. Many works have been published to solve the Duffing equation numerically, especially to analyze the different types of conservatives of the Duffing equation (Nourazar & Mirzabeigy, 2013).

Leung and his team proposed the residue harmonic balance analysis for the damped Duffing resonator driven by a van der Pol oscillator. Their solutions were accurate and in agreement with respect to the numerical integration solutions under different parameters (Leung, Guo, & Yang, 2012).

D. D. Ganji et al. used He's Energy Balance Method (EBM) to solve strong nonlinear Duffing oscillators with cubic-quintic nonlinear restoring forces.

S. Nourazar and A. Mirzabe used the modified differential transform method for solving the nonlinear Duffing oscillator with damping and found that this method agrees with the results from using the fourth-order Runge–Kutta method. There is a very high degree of accuracy in the entire domain even if the amplitude of oscillation reduces over time (Nourazar & Mirzabeigy, 2013).

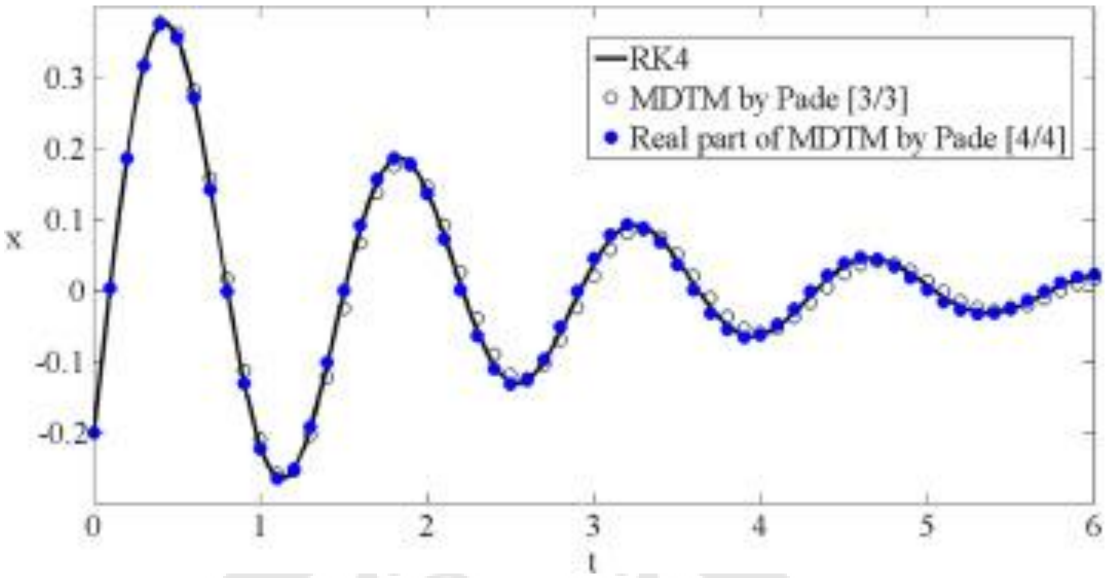

Figure 5 S. Nourazar and A. Mirzabeigy compare their modified differential transform method with the fourth-order Runge–Kutta method

Source: S. Nourazar. (2013). Approximate solution for nonlinear Duffing oscillator with damping effect using the modified differential transform method p. 367.

They show that there are many methods that have been developed for solving the Duffing equation with very high accuracy.

4 Theory of Averaging and Liénard Theorem

As we know the van der Pol equation can be expressed by

$$\frac{d^2x}{dt^2} - \mu(1-x^2)\frac{dx}{dt} + x = 0. \tag{2.18}$$

And we can change it into 2 autonomous equations

$$\frac{dx}{dt} = y - F(x) := y - \mu\left(\frac{x^3}{3} - x\right) \tag{2.19}$$

and

$$\frac{dy}{dt} = -x. \tag{2.20}$$

If $\mu = 0$ the equation will become

$$\frac{d^2x}{dt^2} + x = 0 \tag{2.21}$$

which is the simple harmonic motion. The system is conservative but, in another case, $\mu \neq 0$ the system became non-conservative. Next, we will look for theories that are useful for this research.

There are 2 theories that are worth mentioning for the van de Pol equation, the theory of averaging and the Liénard theorem. We shall start with the theory of averaging first.

The theory of averaging is very important in dynamical systems and perturbation theory. For the van der Pol equation it can be shown that there exists a unique stable limit cycle in the phase space. The theory of averaging will exclude the fast oscillations and show the residual curve. This theory uses time-scale separation.

For theory of averaging, let a compact set $U \subseteq \mathbb{R}^n$ where ϵ satisfied $0 < \epsilon \ll 1.$

If we have z and x are the solutions of these equations,

$$\frac{dx}{dt} = \epsilon f(x, t, \epsilon) \tag{2.22}$$

and

$$\frac{dz}{dt} = \epsilon \bar{f}(z) := \epsilon \frac{1}{T} \int_0^T f(z, t, 0) dt \tag{2.23}$$

respectively, where $f : \mathbb{R}^n \times \mathbb{R} \times \mathbb{R}^+ \rightarrow \mathbb{R}^n$ is C^2 and periodic in t which has period of T, which initial state $x(0) = x_0$and $z(0) = z_0$ are satisfied where $|x_0 - z_0| = O(\epsilon)$, then $|x(t) - z(t)| = O(\epsilon)$on a time scale $t \sim \frac{1}{\epsilon}$.

The averaging principle and theory of averaging has a long history, which first came from the perturbation problems that arose in celestial mechanics (Chicone, 2006). For the case of the van der Pol equation, theory of averaging shows the unique limit cycle of the van der Pol equation. In every initial condition, all other trajectories will approach to this limit cycle.

The Liénard theorem is very important to help understand dynamical systems that have a unique stable limit cycle. The van der Pol equation satisfies this theorem. The theorem is named after Alfred-Marie Liénard (1869-1958), a French physicist. For the proof of a classical result of the uniqueness of the limit cycles for systems.

Oscillating electrical circuits can be described by second order differential equation

$$\frac{d^2x}{dt^2} + f(x)\frac{dx}{dt} + g(x) = 0. \qquad (2.24)$$

This (2.24) equation is called a Liénard's equation. From the van der Pol equation we can generalize it to a Liénard's equation. For Liénard's equation we can change it to a system of 2 equations.

$$\frac{dx}{dt} = y \qquad (2.25)$$

and

$$\frac{dy}{dt} = -g(x) - f(x)y. \qquad (2.26)$$

Liénard theorem told us that if we put the fitting assumption f and g we can conclude that the system has a stable and unique limit cycle. We can write down the Liénard theorem as

suppose that $f(x)$ and $g(x)$ satisfy the following conditions:

- $f(x)$ and $g(x)$ are continuously differentiable for all values of x.
- $g(-x) = -g(x)$ for all x (which means $g(x)$ is an odd function.)
- $g(x) > 0$ for $x > 0$.
- $f(-x) = f(x)$ for all (which means $f(x)$ is an even function.)
- The odd function $F(x) = \int_0^x f(u)du$ has exactly one positive zero at $x = a,$ is negative for $0 < x < a,$ is positive and nondecreasing for $x > a,$ and $F(x) \to \infty$ as $x \to \infty$.

If all the 5 conditions are satisfied, we can conclude that system (2.26) has a stable and unique limit cycle around the initial point of the plotted phase plane analysis (Strogatz, 2015).

Corollary: The van der Pol equation, with $\mu > 0$, has a unique stable limit cycle.

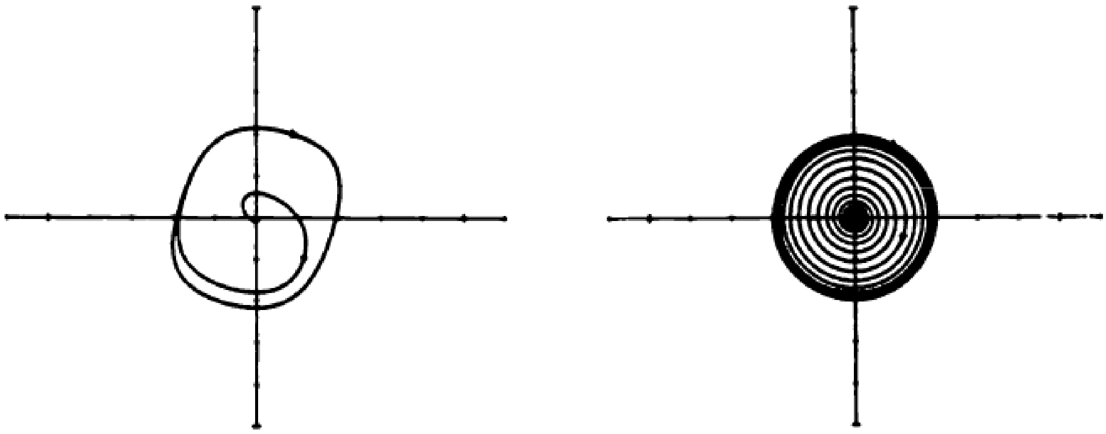

Figure 6 The limit cycle for the van der Pol equation for $\mu = 1$ and $\mu = 0.1$

Source: L. Perko. (2002). Differential equations and dynamical systems p. 257.

5 Ordinary Differential Equation

The van der Pol oscillator is described by a differential equation. This is a second order differential equation with non-linear damping. Differential equations play a key role in solving many problems over a very broad range of disciplines such as physical science, engineering, biological science and also economics.

In physical science, especially mechanics, the fundamental subject in physics, we write Newton's second law of motion in the form of a differential equation called the equation of motion. This equation describes the motion of the system as a function of time.

In mechanics we also have the Euler-Lagrange equations, or simply just "Lagrange's equation". This provides a more sophisticated description of the system. Lagrange's equation is a second order partial differential equation. We use this in Lagrangian mechanics.

In thermodynamics, we also use differential equations such as the heat equation and Newton's law of cooling. In fluid dynamics we use the convection-diffusion equation and the Navier-Stokes equation.

In economics we have the Malthusian growth model, in chemistry we have the rate equation and in biology we have the predator and prey equation and the Hodgkin-Huxley model. So, we can conclude that differential equations are very important for understanding how things work.

In history differential equations have come together with the discovery of calculus by Newton that may have been influenced by Isaac Barrow, his teacher from Isaac Barrow's Geometrical Lectures (Kitcher, 1973).

Newton completed a book entitled “Method of Fluxions” in 1671. In this book, Newton writes about fluxions, or as we know them today, derivatives. In this book Newton first introduced the three differential equations.

$$\frac{dy}{dx} = f(x) \tag{2.27}$$

$$\frac{dy}{dx} = f(x, y) \tag{2.28}$$

$$x_1 \frac{\partial y}{\partial x_1} + x_2 \frac{\partial y}{\partial x_2} = y. \tag{2.29}$$

The first two equations are ordinary differential equations and the last one is a partial differential equation.

Some studies show that the differential equation was first introduced by Leibniz in 1675 when he wrote

$$\int x\, dx = \frac{1}{2}\, x^2. \tag{2.30}$$

In 1692, with help from Leibniz, James Bernoulli discovered a method of integrating some homogeneous differential equations of the first order.

In 1690 James Bernoulli thought about the problem of the Isochrone. He published the solution in “Acta eruditorum” this problem introduced a differential equation expressing the equality of two differentials. This showed that the integrals of the two members of the equations were equal. In the past this was called calculus summatorius or as we know it today as integral calculus.

In the 18th century there were problems concerning the vibrating sting in musical instruments that used the wave equation for a solution. The wave equation is a second order linear partial differential equation. We can conclude that in more than 400 years since Newton, we use differential equations to solve many problems. Not every problem involves a differential equation that can be solved analytically, so we had to develop many numerical methods to solve these problems numerically.

6 Numerical Methods for Ordinary Differential Equations

We cannot solve some mathematical problems analytically, but we can solve them numerically. Numerical analysis uses numerical approximation to solve problems. There is a wide range of methods to find numerical approximations. Since the mid-20th century, the speed and power of computing technology has increased dramatically. Scientists and researchers can implement numerical algorithms to obtain solutions.

There are some concerns with numerical approximation. There are two sources of error in a solution. First the numerical method involves analytical approximations. Second, an algorithm may be unstable due to numerical representation and rounding errors. We must care about the acceptable bound of error in the solutions.

Round-off error commonly occurs in numerically analysis. For example, Kaneko and Liu have published a study entitled "Accumulation of Round-Off Error in Fast Fourier Transforms" (Kaneko & Liu, 1970). This study shows that the finite word length used in the fast Fourier transform (FFT) in the computer causes an error in Fourier coefficients. This research derives explicit expressions for the mean square error in the fast Fourier transform when floating-point arithmetic is used. Rounding error happens because not all real numbers can be written exactly in the computer memory where information storage is limited. For example, we cannot write $\sqrt{2}$ in an exact form in the computer. In some cases, we must avoid accumulated propagated error that increases with time.

Table 1 Examples of round-off error

Number	Representation	Approximation	Error
2	2.0	2.0	0
$\sqrt{2}$	1.414 213 562 37…	1.414 213	0.000 000 562 37…
$\sqrt{3}$	1.732 050 807 57…	1.732 050	0.000 000 807 57…

Another well-known error is truncation error. J. H. Verner has published a study entitled "Explicit Runge–Kutta Methods with Estimates of the Local Truncation Error" (Verner, 1978). This study uses a new method that is better than Fehlberg's method. Approximations of the solutions of ordinary differential equations rely on controlling estimates of the error through adjustment of stepsize. Explicit Runge-Kutta methods estimates of the local truncation error are normally used. Fehlberg's method gives error estimates equally zero when it is a quadrature problem so the estimate is not reliable. This study tries to use better methods with higher accuracy.

Truncation error can be understood by using a Taylor series given by

$$f(x) = f(a) + f'(a)(x-a) + \frac{f''(a)}{2!}(x-a)^2 + \frac{f^3(a)}{3!}(x-a)^3 + \cdots + \frac{f^n(a)}{n!}(x-a)^n + \cdots \tag{2.31}$$

For example.

$$\ln x = \ln a + \frac{x-a}{a} - \frac{(x-a)^2}{2a^2} + \frac{(x-a)^3}{3a^3} - \frac{(x-a)^4}{4a^4} + \frac{(x-a)^5}{5a^5} - \cdots . \tag{2.32}$$

for $|x-a| < 1$.

Truncation error happens when an infinite sum is approximated by a finite sum. For example

$$\ln x \cong \ln a + \frac{x-a}{a} - \frac{(x-a)^2}{2a^2} + \frac{(x-a)^3}{3a^3}, \tag{2.33}$$

and the truncation error is

$$\frac{(x-a)^4}{4a^4} + \frac{(x-a)^5}{5a^5} - \cdots. \qquad (2.34)$$

Discretization error is also important and arises from trying to compute using a finite number of steps for a numerical approximation to an infinite process.

Numerical analysis is used in many branches of mathematics such as ordinary differential equations where we have Euler's method as the simplest. Interpolation, the method of constructing new data points in the range by estimating the new value by extending a known sequence of values, includes linear interpolation as the simplest case. Extrapolation, estimating the new value beyond an original specific range of given values, is also used. For numerical integration we have a trapezoidal rule as the simplest case. For root finding, finding the solution(s) of a nonlinear equation, we have the Newton–Raphson method.

Recently the trend of research in numerical analysis is to try to improve methods for better accuracy, less error and more efficient execution such as Tatang's new method for numerical geophysical models (Tatang, Pan, Prinn, & McRae, 1997) and Kodera for a new method for non-stationary signals (Kodera, De Villedary, & Gendrin, 1976).

There are many numerical methods for differential equations. Some have pros, some have cons. These methods are used to find numerical approximations for the solution of differential equations. Partial differential equations can also be solved by numerical methods. The numerical methods for differential equations used in this research will be discussed briefly.

6.1 Euler Method

We should discuss this method first because it is the simplest method for approximation of the solution of a differential equation. This method has drawbacks because a large error is accumulated when the process proceeds, although it is easy to understand the error analysis of this method. If we want a more accurate solution, we should use another more sophisticated numerical method. Euler's method is explained in the book “Institutionum calculi integralis” by Leonhard Euler published in 1768. Euler’s method can solve initial value problems (IVP) explicitly.

An initial value problem is stated $y' = f(t, y)$ with data $y(a) = y_0$. Let $[a, b]$ be the interval over which we want to find the solution. We need to construct a set of points where we calculate an approximate solution. We subdivide the interval $[a, b]$ into M steps between points.

$$t_k = a + kh \text{ for } k = 0,1, \dots, M. \tag{2.35}$$

The step side is $h = \frac{b-a}{M}$.

We solve the initial value problem approximately

$$y' = f(t, y) \text{ over } [t_0, t_M] \text{ with } y(t_0) = y_0. \tag{2.36}$$

We use a Taylor series to expand $y(t)$ and then we will have, for the first step,

$$y(t_1) = y\,(t_0) + hf\left(t_{0,}\, y(t_0)\right) + y''(c_1)\frac{h^2}{2}. \tag{2.37}$$

The Euler approximation is

$$y_1 = y_0 + hf(t_0, y_0). \tag{2.38}$$

This process is repeated step by step and generates a sequence of values that approximate the solution curve y=y(t). So, the Euler method is

$$t_{k+1} = t_k + h,\ \ y_{k+1} = y_k + hf(t_k, y_k), \tag{2.39}$$

for $k = 0,1, \dots, M-1$ (Mathews & Fink Kurtis, 1999).

Example of Euler method

Suppose $\frac{dy}{dx} = \cos(x + y) - e^x$ (2.40)

$y(0) = 2$

$h = 0.1$

We start at initial value (0,2) to calculate the value of the derivative at this point so.

$$\begin{aligned}\frac{dy}{dx} &= \cos(x + y) - e^x \\ &= \cos(2) - 1\end{aligned}$$

$$= -1.4161468365$$

We substitute our starting point and the derivative to obtain the next point along

$$y(x+h) \approx y(x) + hf\ (x,y)$$
$$y\ (0.1) \approx 2 + 0.1(-1.4161468365)$$
$$\approx 1.85838531635$$

Next, we need to calculate the value of the derivative at the new point (0.1,1.85838531635)

$$\frac{dy}{dx} = \cos(x+y) - e^{x}$$
$$= \cos(0.1 + 1.85838531635) - e^{0.1}$$
$$= -1.483128264943$$

Next, we substitute our current and the derivative to obtain the next point along.

$$y(x+h) \approx y(x) + hf(x,y)$$
$$y(0.2) \approx 1.85838531635 + 0.1(-1.483128264943)$$
$$\approx 1.7100724898557$$

From the Euler method we can make a table of solution.

Table 2 A table of solution from Euler method

X	Y	$\frac{dy}{dx}$
0	2	-1.4161468365
0.1	1.85838531635	-1.483128264943
0.2	1.7100724898557	-1.4584022320

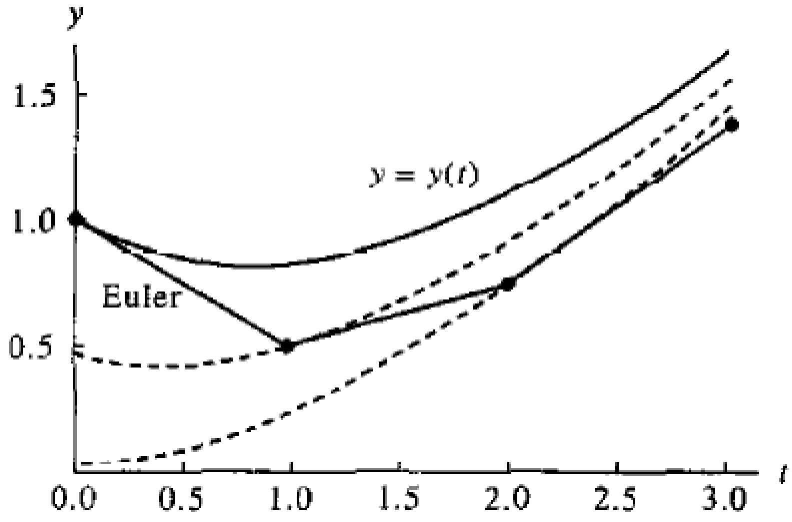

Figure 7 Euler's method approximation $y_{k-1} - y_k + hf(t_k, y_k)$

Source: J. H. Mathews. (1999). Numerical methods using Matlab p. 436.

6.2 Heun's Method

This method is named after Karl Heun (1859-1929), a German mathematician. His contributions are mostly in the field of differential equations. Heun's method is better than Euler's method for solving initial value problems in differential equations. The more sophisticated method, Runge-Kutta methods contain an appreciation of the early work included in Heun's method (Butcher & Wanner, 1996).

We used Heun's method to solve initial value problems

$$y'(t) = f\big(t, y(t)\big) \text{ over } [a, b] \text{ with } y(t_0) = y_0. \qquad (2.41)$$

We can use the fundamental theorem of calculus, the theorem link between differentiate and integrate of the function, to integrate $y'(t)$ over $[t_0, t_1]$ to obtain the solution point (t_1, y_1)

$$\int_{t0}^{t1} f(t, y(t)dt = \int_{t0}^{t1} y'(t)dt = y(t_1) - y(t_0). \qquad (2.42)$$

This equation (2.42) is solved for $y(t_1)$ the result will be

$$y(t_1) = y(t_0) + \int_{t0}^{t1} f\big(t, y(t)\big)dt. \qquad (2.43)$$

We can approximate the definite integral (2.43) by the trapezoidal rule so $y(t_1)$ will be

$$y(t_1) \approx y(t_0) + \frac{h}{2}\Big(f\big(t_0, y(t_0)\big) + f\big(t_1, y(t_1)\big)\Big). \quad (2.44)$$

If we look at the right-hand side of this equation, we will see $y(t_1)$. We can use Euler's method to determine $y(t_1)$. The equation will change to

$$y_1 = y(t_0) + \frac{h}{2}\begin{pmatrix} f(t_0, y_0) \\ +f\big(t_1, y_0 + hf(t_0, y_0)\big) \end{pmatrix}. \quad (2.45)$$

This is the Heun's method for finding (t_1, y_1).

This process will repeat again and again to create a sequence of points to approximate the solution curve $y = y(t)$. For each step Euler's method will be used for a prediction and the trapezoidal rule will be used for a correction to obtain the final value. Finally, the general steps for Heun's method are:

$$P_{k+1} = y_k + hf(t_k, y_k), \qquad t_{k+1} = t_k + h. \quad (2.46)$$

And

$$y_{k+1} = y_k + \left(\frac{h}{2}\right)\big(f(t_k, y_k) + f(t_{k+1}, p_{k+1})\big). \quad (2.47)$$

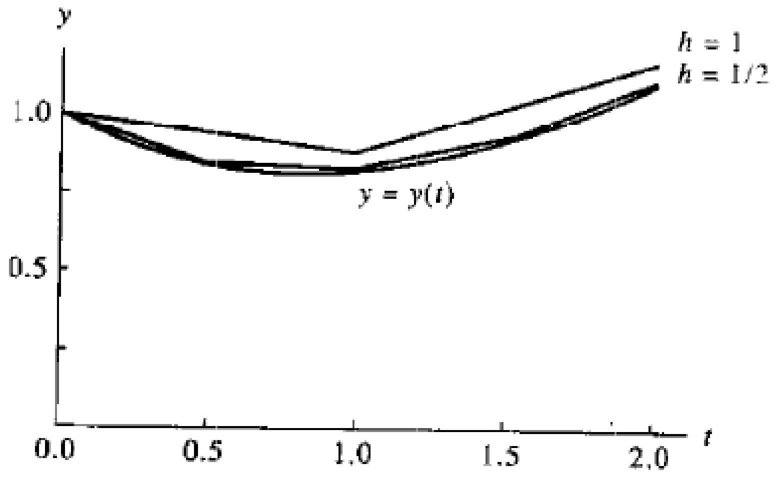

Figure 8 Comparison of the solutions of Heun's method with different step side $h = 1$ and $h = \frac{1}{2}$

Source: J. H. Mathews. (1999). Numerical methods using Matlab p. 446.

The Euler method's accumulated error is $O(h^1)$ compared to the Heun's method which after M steps has accumulated error

$$-\sum_{k=1}^{M} y^{(2)}(c_k)\left(\frac{h^3}{12}\right) \approx \frac{b-a}{12}\, y^{(2)}(c)h^2 = O(h^2). \quad (2.48)$$

This compares favorably with the Euler method since $h^2 \ll h$.

6.3 The Runge-Kutta Methods

This method has proposed in 1901 By Martin Kutta (1867-1944) and Carl Runge (1856-1927). Both were German mathematicians. They were interested in mathematical physics. In comparison, this method gives a more accurate approximation than Euler's method and Heun's method. Each Runge-Kutta method is derived from Taylor expansions. The most well-known Runge-Kutta method is the fourth-order Runge-Kutta method, simply known as RK4. The fourth-order Runge-Kutta method is the best for programming because it is quite accurate, not complicated and very stable. RK4 is based on computing y_{k+1} as

$$y_{k+1} = y_k + w_1 k_1 + w_2 k_2 + w_3 k_3 + w_4 k_4. \qquad (2.49)$$

Where

$$k_1 = hf\ (t_k, y_k)$$
$$k_2 = hf\ (t_k + a_1 h, y_k + b_1 k_1)$$
$$k_3 = hf\ (t_k + a_2 h, y_k + b_2 k_1 + b_3 k_2)$$
$$k_4 = hf(t_k + a_3 h, y_k + b_4 k_1 + b_5 k_2 + b_6 k_3).$$

The local truncation error is of order $O(h^5)$. The parameters are given in the following system of equations

$$b_1 = a_1$$
$$b_2 + b_3 = a_2$$
$$b_4 + b_5 + b_6 = a_3$$
$$w_1 + w_2 + w_3 + w_4 = \frac{1}{1}$$
$$w_2 a_1 + w_3 a_2 + w_4 a_3 = \frac{1}{2}$$
$$w_2 a_1^2 + w_3 a_2^2 + w_4 s_3^2 = \frac{1}{3}$$
$$w_2 a_1^3 + w_3 a_2^3 + w_4 a_3^3 = \frac{1}{4}$$
$$w_3 a_1 b_3 + w_4 (a_1 b_5 + a_2 b_6) = \frac{1}{6}$$
$$w_3 a_1 a_2 b_3 + w_4 a_3 (a_1 b_5 + a_2 b_6) = \frac{1}{8}$$
$$w_3 a_1^2 b_3 + w_4 (a_1^2 b_5 + a_2^2 b_6) = \frac{1}{12}$$
$$w_4 a_1 b_3 b_6 = \frac{1}{24}.$$

This system of equations involves 11 equations with 13 unknowns. To solve these equations, we take

$$a_1 = \frac{1}{2} \quad , b_2 = 0.$$

The solution for the remaining variables is

$$a_2, b_1, b_3 = \frac{1}{2}\ , \qquad a_3, b_6, = 1$$
$$b_4, b_5 = 0\ , \qquad w_1, w_4 = \frac{1}{6}$$
$$w_2, w_3 = \frac{1}{3}.$$

Substitute these variables into the equations and we will obtain the fourth order Runge-Kutta. Start with the initial point (t_0, y_0) and generate the sequence of the approximation when

$$y_{k+1} = y_k + \frac{h(f_1+2f_2+2f_3+f4)}{6}. \tag{2.50}$$

Which

$$f_1 = f(t_k, y_k) \tag{2.51}$$

$$f_2 = f\ (t_k + \frac{h}{2}, y_k + \frac{h}{2} f_1) \tag{2.52}$$

$$f_3 = f\left(t_k + \frac{h}{2}, y_k + \frac{h}{2} f_2\right) \tag{2.53}$$

$$f_4 = f(\ t_k + h, y_k + hf_3). \tag{2.54}$$

6.4 Runge-Kutta-Fehlberg Method

Erwin Fehlberg developed this method further based on Runge-Kutta method and he published this method in 1969. Fehlberg was a German mathematician. This method is also called RKF45. It is a numerical method for differential equations with an adaptive stepsize to be determined automatically. In RKF45 each step is required to use the following six values that include

$$k_1 = hf\ (t_k, y_k) \tag{2.55}$$

$$k_2 = hf\ \left(t_k + \frac{1}{4}h, y_k + \frac{1}{4}k_1\right) \tag{2.56}$$

$$k_3 = hf\ (t_k + \frac{3}{8}h, y_k + \frac{3}{32}k_1 + \frac{9}{32}k_2 \tag{2.57}$$

$$k_4 = hf\ \left(t_k + \frac{12}{13}h, y_k + \frac{1932}{2197}k_1 - \frac{7200}{2197}k_2 + \frac{7296}{2197}k_3\right) \tag{2.58}$$

$$k_5 = hf\ \left(t_k + h, y_k + \frac{439}{216}k_1 - 8k_2 + \frac{3680}{513}k_3 - \frac{845}{4104}k_4\right) \tag{2.59}$$

$$k_6 = hf\ \left(t_k + \frac{1}{2}h, y_k - \frac{8}{27}k_1 + 2k_2 - \frac{3544}{2565}k_3 + \frac{1859}{4104}k_4 - \frac{11}{40}k_5\right). \tag{2.60}$$

6.5 The Adams-Bashforth-Moulton Methods

Unlike Euler's method which is a single-step method, it uses only the initial point (t_0, y_0) to compute (t_1, y_1). When we have the initial point then it takes a short step forward in time to find the next solution point, continues with subsequence until the solution is reached. Euler's method uses the only previous point and its derivative to determine the current value. The Adams-Bashforth-Moulton methods, on the other hand, use a linear multistep method which uses the information from the previous steps. This means more efficiency and accuracy of the answers at each step. Using the combinations of a predictor and corrector requires only just two function evaluations of $f(t, y)$ at each step.

The Adams-Bashforth-Moulton method is named after John Couch Adams (1819-1892) a British mathematician, Francis Bashforth (1819-1912) a British mathematician and Forest Ray Moulton (1872-1952) an American scientist. The Adams-Bashforth-Moulton method is a Predictor-Corrector method with linear multistep method for solving differential equations numerically. It is derived from the fundamental theorem of calculus.

$$y(t_{k+1}) = y(t_k) + \int_{t_k}^{t_{k+1}} f\big(t, y(t)\big)dt. \tag{2.61}$$

Then use the Lagrange polynomial approximation as the predictor for $f\big(t, y(t)\big)$ based on the points $(t_{k-3}, f_{k-3}), (t_{k-2}, f_{k-2}), (t_{k-1}, f_{k-1})$ and (t_k, f_k). Then it is integrated over $[t_k, t_{k+1}]$ in (2.61). The Adams-Bashforth-Moulton methods have been produced from this process,

$$p_{k+1} = y_k + \frac{h}{24}(-9f_{k-3} + 37\, f_{k-2} - 59\, f_{k-1} + 55f_k). \tag{2.62}$$

Besides the predictor we also have the corrector, which is developed similarly. The value of p_{k+1}that we have previously will be used for now. We construct a second Lagrange polynomial for $f\big(t, y(t)\big)$ which is based on the points $(t_{k-2}, f_{k-2}), (t_{k-1}, f_{k-1}), (t_k, f_k)$.

And the new point equal

$$(t_{k+1}, f_{k+1}) = \big(t_{k+1}, f(t_{k+1}, p_{k+1})\big). \tag{2.63}$$

This polynomial is then integrating over $[t_k, t_{k+1}]$ and finally it will produce the Adams-Moulton corrector

$$y_{k+1} = y_k + \frac{h}{24}(f_{k-2} - 5f_{k-1} + 19f_k + 9f_{k+1}). \quad (2.64)$$

7 Non-linearity in the Circuit Systems

In the van der Pol oscillator,

$$\frac{d^2x}{dt^2} - \mu(1 - x^2)\frac{dx}{dt} + x = 0. \quad (2.65)$$

The van der Pol oscillator in electrical circuits is made up of 3 components: inductor (L), capacitor (C) and a non-linear resistance (R). Van der Pol found an equation that can model the oscillations of a self-maintained electrical circuit. The intensity-tension U_R of the non-linear resistance (R) is

$$U_R = -R_0 i_0 \left(\frac{i}{i_0} - \frac{1}{3}\left(\frac{i}{i_0}\right)^3\right). \quad (2.66)$$

Where U_R, R_0, i_0, i are intensity-tension of the non-linear resistance, resistance of the normalization, current of the normalization and current respectively.

We can use the operational amplifier (op-amp) to obtain the non-linear resistance. From link's law we will have

$$U_L + U_R + U_C = 0. \quad (2.67)$$

Where U_L, U_R, U_C are tension to the limit of the inductor, intensity-tension of the non-linear resistance and tension to the limit of the capacitor respectively which

$$U_L = L\frac{di}{d\tau}, \quad (2.68)$$

$$U_C = \frac{1}{C}\int i d\,\tau. \quad (2.69)$$

Substituting we have

$$L\frac{di}{d\tau} - R_0 i_0 \left(\frac{i}{i_0} - \frac{1}{3}\left(\frac{i}{i_0}\right)^3\right) + \frac{1}{C}\int i d\,\tau = 0. \quad (2.70)$$

Then differentiate with respect to τ, we will have

$$L\frac{d^2 i}{d\tau^2} - R_0\left(1 - \frac{i^2}{i_0^2}\right)\frac{di}{d\tau} + \frac{i}{C} = 0. \quad (2.71)$$

For convenience we set

$$x = \frac{i}{i_0}, \quad t = \omega_e \tau. \tag{2.72}$$

Where $\omega_e = \frac{1}{\sqrt{LC}}$ is an electric pulsation, we have,

$$\frac{d}{d\tau} = \omega_e \frac{d}{dt}, \tag{2.73}$$

$$\frac{d^2}{d\tau^2} = \omega_e^2 \frac{d^2}{dt^2}. \tag{2.74}$$

Substituting and finally we will have

$$\frac{d^2x}{dt^2} - R_0\sqrt{\frac{C}{L}}\,(1 - x^2)\frac{dx}{dt} + x = 0. \tag{2.75}$$

So, we will have the van der Pol equation

$$\frac{d^2x}{dt^2} - \mu(1 - x^2)\frac{dx}{dt} + x = 0, \tag{2.76}$$

we set $\mu = R_0\sqrt{\frac{C}{L}}$.

μ is a scalar (not vector) parameter showing the property of the strength of the non-linear damping in a self-maintained electrical circuit (Hafeez, Ndikilar, & Isyaku, 2015).

In 1978 Yoshisuke Ueda published a paper in the "International Journal of Non-Linear Mechanics" about the non-linearity in the system described by the Duffing equation. The random oscillations which occur in a series-resonance circuit containing a saturable inductor under the impression of a sinusoidal voltage, can be written as

$$n\frac{d\phi}{dt} + Ri_R = E\sin(\omega t) \tag{2.77}$$

$$Ri_R = \frac{1}{C}\int i_c dt \tag{2.78}$$

and

$$i = i_R + i_c. \tag{2.79}$$

When we substitute some variables and eliminating i_R and i_c, the equation will become

$$\frac{d^2x}{d\tau} + k\frac{dx}{d\tau} + x^3 = B\cos(\tau), \tag{2.80}$$

which is the Duffing equation

where

$$\tau = \omega t - \tan^{-1} k \tag{2.81}$$

$$k = \frac{1}{\omega CR} \tag{2.82}$$

and

$$B = \frac{E}{n\omega\Phi_n}\sqrt{1+k^2}. \tag{2.83}$$

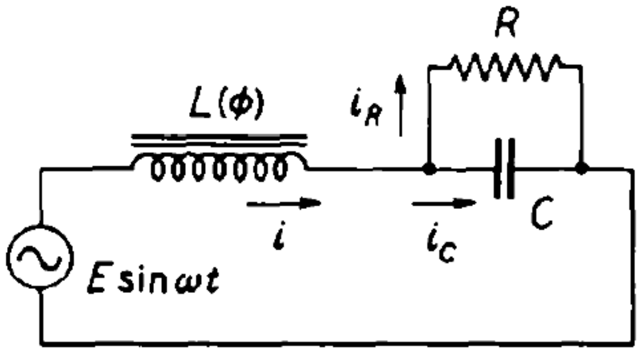

Figure 9 Ueda's series-resonance circuit with non-linear inductance

Source: Y. Ueda. (1978). Chaotically transitional phenomena in the forced negative-resistance oscillator p. 481.

By using analog and digital computers Ueda observed the random oscillation in waveforms, phase-plane analysis, spectral analysis, and spectral decomposition of the power. He found that the random oscillation was not a special one, which appeared only for particular values but could be noticed at different parameters. He concluded that the random oscillation was the representative point of the actual system continues to transit randomly among the infinitely many solutions due to the perturbations by uncertain factors of the system. The other thing he noted was the average power spectrum of the random oscillation depends practically not on the nature of uncertain factors but on the structure of the solutions emanating from the attractor (Ueda, 1978).

In 1979 Yoshisuke Ueda published a paper about Duffing equation (Ueda, 1979),

$$\frac{d^2x}{dt^2} + k\frac{dx}{dt} + f(x) = e(t). \tag{2.84}$$

Where $e(t)$ is a periodic function of the period 2π.

The uncertain factor in the physical system for a small random phenomenon can sometimes be neglected but sometimes it cannot. This phenomenon was call turbulent or chaotic behavior. Ueda used a computer simulation for the electric circuit with non-linear inductance under a sinusoidal voltage governed by the Duffing equation. For a randomly transitional process we use a specific case of the Duffing equation,

$$\frac{d^2x}{dt^2} + k\frac{dx}{dt} + x^3 = B\cos(t). \tag{2.85}$$

Which we can transform into 2 first order differential equation

$$\frac{dx}{dt} = y \tag{2.86}$$

and

$$\frac{dy}{dt} = -ky - x^3 + B\cos(t). \tag{2.87}$$

We have to use a bundle of solution that have asymptotically stable rather than a single solution of the equation to represent the chaotic behavior of the system.

$$\lambda = (k, B) = (0.1, 12.0) \tag{2.88}$$

will be a deterministic process or randomly transitional process depending on the initial conditions.

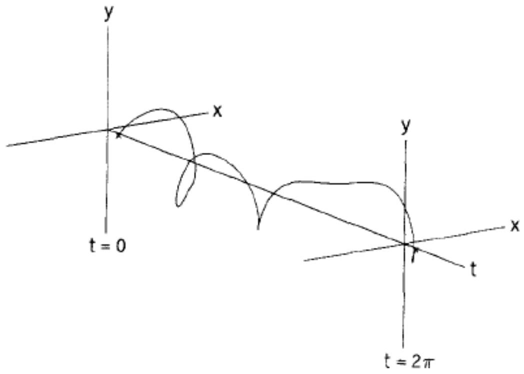

Figure 10 The periodic solution representing the deterministic process in txy space

Source: Y. Ueda. (1979). Randomly transitional phenomena in the system governed by Duffing's equation p. 183.

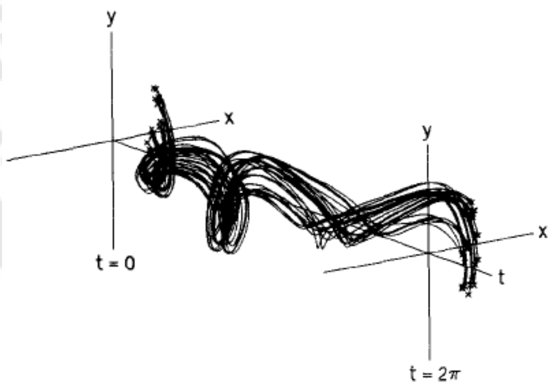

Figure 11 The bundle of solutions representing the random process in txy space

Source: Y. Ueda. (1979). Randomly transitional phenomena in the system governed by Duffing's equation p. 183.

From a Duffing equation, Ueda showed the dependence of the attractors on the system parameters by keeping k= 0.1 to be a constant and varied the value of B.

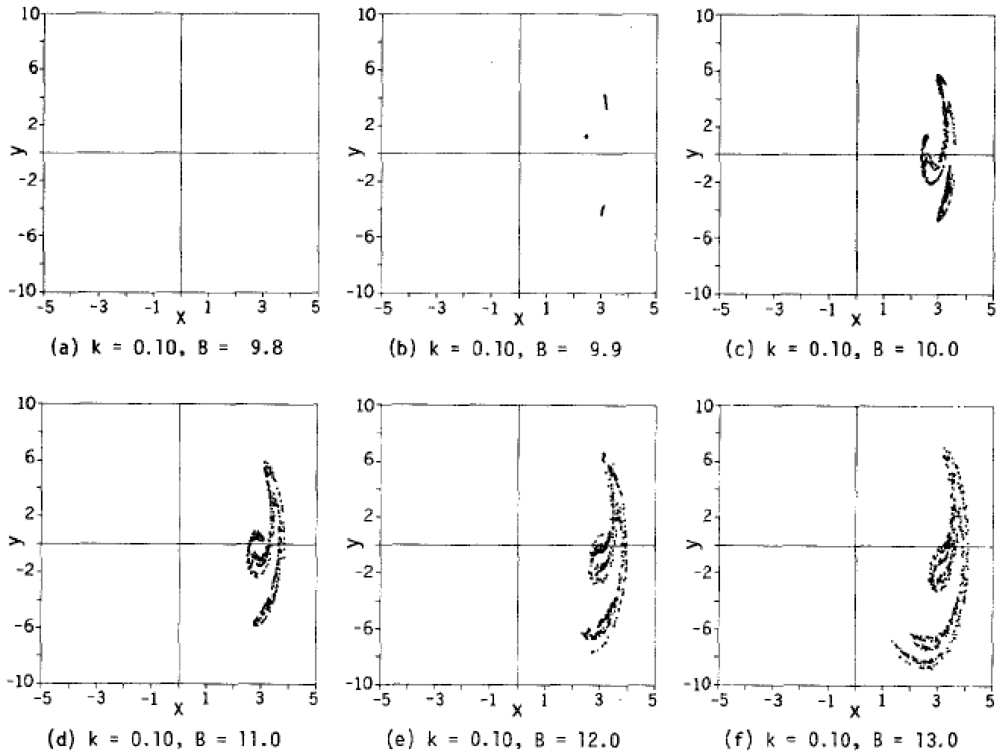

Figure 12 The attractors of the system which varied the value of B

Source: Y. Ueda. (1979). Randomly transitional phenomena in the system governed by Duffing's equation p. 187.

In this paper Ueda obtained the dependence of the process on the system parameters by showing the transition of the attractors and the average power spectra (Ueda, 1978).

In 1981 Yoshisuke Ueda studied the chaotically transitional phenomena in the forced negative-resistance oscillator. He introduced the equation systems describing the forced negative-resistance oscillator by

$$\frac{dx}{d\tau} = y \tag{2.89}$$

$$\frac{dy}{d\tau} = \mu(1 - x^2)y - x^3 + B\cos(\nu\tau) \tag{2.90}$$

which derived from

$$L\frac{di}{dt} + Ri + v = E\cos(\omega t) \tag{2.91}$$

where

$$i_1 = C\frac{dv}{dt} \tag{2.92}$$

and

$$i_2 = f(v) = -Sv\left(1 - \frac{v^2}{V_s^2}\right) \tag{2.93}$$

which

$$i = i_1 + i_2. \tag{2.94}$$

Ueda showed the difference between the almost periodic oscillations and the chaotically transitional process. He showed the results with different parameter as in table 3 (Ueda, 1978).

Table 3 Ueda's results with the different parameters

B	v	Total power	Periodic component	Chaotic component	Remarks
0	1.617	1.67	1.67	0	Self-excited oscillation
1.0	4.0	1.66	1.66	0	Almost periodic oscillation
17.0	4.0	2.74	1.34	1.40	Chaotically transitional process
1.0	0.94	1.01	0.29	0.72	Chaotically transitional process
1.2	0.92	1.02	0.40	0.62	Chaotically transitional process
2.4	0.70	1.24	0.90	0.34	Chaotically transitional process
2.0	0.60	1.04	0.71	0.33	Chaotically transitional process

8 Related Research and Applications

There are many research papers about the van der Pol equation. We will select some papers that are related to this research and their details will be discussed briefly.

There is no general theory for integrating a nonlinear ordinary differential equation. The derivation of the exact solutions for nonlinear ordinary differential equations is non-trivial. Humi et al. show that they can use a generalization of the Cole-Hopf transformation (Humi, 2013a) to linearize and solve various nonlinear ordinary differential equation including the Duffing equation and the perturbed van der Pol equation (Van der Pol equation with additional quadratic and cubic linear term) without forces (Humi, 2013b).

Cordshooli and Vahidi use another method to solve the Duffing-van der Pol equation by using Adomian's decomposition method for second order nonlinear differential equations. This method turns the second order equation into a system of first order differential equations and finally solves them by Adomian's decomposition method again. Many researchers published papers with modifications of Adomian's decomposition method. Adomian's decomposition method works by splitting the equation into linear and nonlinear parts. After that, inverting the higher order derivative operator contained in the linear operator on both sides, and then identifying the initial and/or boundary conditions and the terms involved with the independent variable alone as an initial approximation. Finally, they have decompose the nonlinear function in terms of Adomian's polynomials and find the successive terms of the series solution by a recurrence relation (Cordshooli & Vahidi, 2011).

In 1987 Qiu and Filanovsky published a research paper showing an analytic approach for a periodic solution of the van der Pol equation with moderate values of damping coefficient which used a harmonic balance method and perturbation techniques for $0 \leq \mu \leq 3$ (Qiu & Filanovsky, 1987).

Applications of the van der Pol equation will be discussed to emphasize the significance of this equation. We will survey the research papers that use the van der Pol equation for its applications. The van der Pol equation is used in many areas of science such as in mathematics, physics and engineering.

After van der Pol introduced his equation in 1920, there have been many research papers about the van der Pol equation and the concept of relaxation oscillations, which become a cornerstone of geometric singular perturbation theory.

In 1961 Fitzhugh used the van der Pol equation to describe a relaxation oscillator generalized by the addition of terms to produce a pair of non-linear differential equations. This model gave either a stable singular point or a limit cycle to describe impulses and physiological states in theoretical models of the nerve membrane (FitzHugh, 1961).

In 1999 Cartwright and his team used the van der Pol-FitzHugh-Nagumo model for excitable media with elastic coupling to model the plates in a geological fault which is useful in seismology (Cartwright, Eguíluz, Hernández-García, & Piro, 1999).

CHAPTER 3

METHODOLOGY

In this chapter the research methodology will be discussed. In this research we used Fortran 90 programming language, the version developed from earlier FORTRAN that first appeared in 1957 and then FORTRAN 77. Fortran is a general-purpose high-level programming language designed for scientific numerical computations. It has been widely used up to the present by scientists and engineers. Fortran was first developed by John Backus when he joined IBM, the American information technology company. He and his team developed Fortran for the IBM 704 computer.

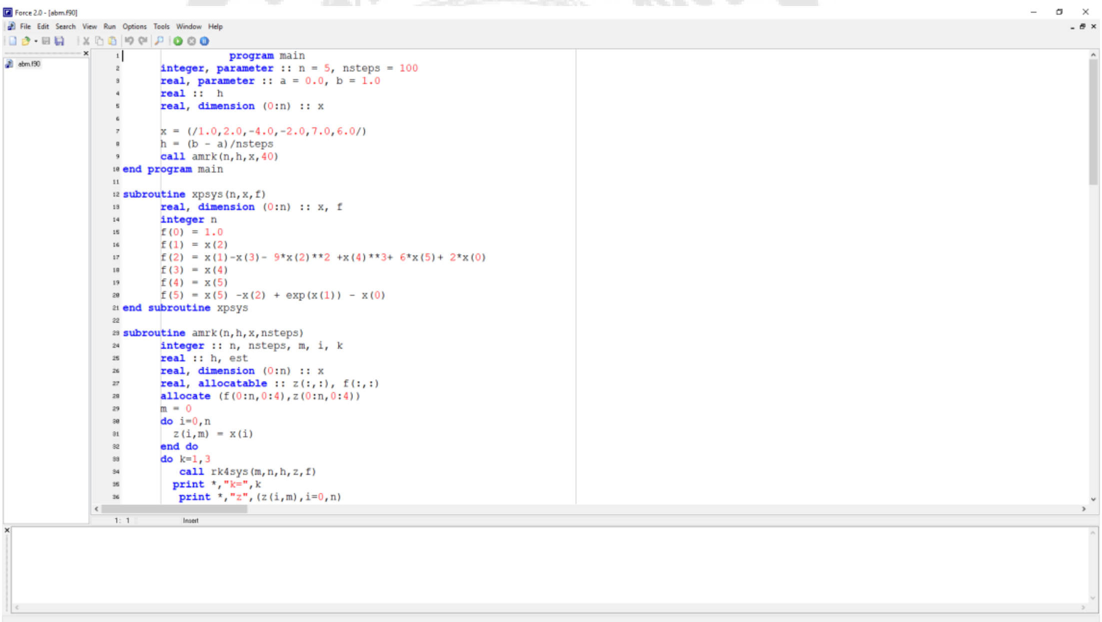

Figure 13 FORCE 2.0, A Fortran complier and editor were used in this research

Suppose we have an electrical circuit which is a negative-resistance with periodic excitation and it has been inserting a periodic excitation $E\cos(\omega t)$ into the electrical circuit.

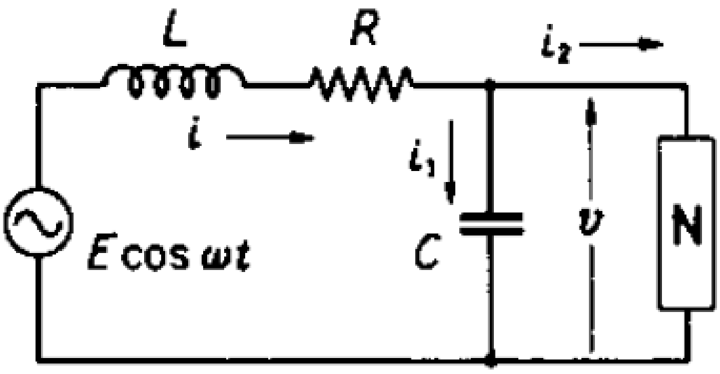

Figure 14 An electrical circuit which is a negative-resistance with periodic excitation

Source: Y. Ueda. (1981). Chaotically transitional phenomena in the forced negative-resistance oscillator p. 217.

Negative resistance is a property in some electrical circuits. For a normal electrical circuit, a resistor obeys the Ohm's law. When resistor R is a constant, the current decreases when the voltage decreases. In the case of a negative resistance circuit, the current is increases when the voltage decreases. Tunnel diodes and gas discharge tubes are examples of negative resistance electrical circuits.

From figure 14 periodic excitation comes from AC voltage source $E\cos(\omega t)$. The path of current i flows through inductor L and resistor R. The current split up flows through different branches. Current i_1 flows through capacitor C and current i_2 flows through a negative-resistance N.

An electrical circuit has AC voltage source $E\cos(\omega t)$, Inductor (L), Resistor (R), and Capacitor (C). We can write these variables into the differential equation,

$$L\frac{di}{dt} + Ri + v = E\cos(\omega t), \tag{3.1}$$

when

$$i_{total} = i_1 + i_2, \tag{3.2}$$

$$i_1 = C\frac{dv}{dt}, \tag{3.3}$$

and

$$i_2 = f(v) = -Sv\left(1 - \frac{v^2}{V_s^2}\right) \quad (3.4)$$

so

$$i_{total} = C\frac{dv}{dt} - Sv\left(1 - \frac{v^2}{v_s^2}\right). \quad (3.5)$$

Which is $S = 1/R$ and V_s is a constant and we also have these variables

$$x = \sqrt{\gamma}\frac{v}{V_s},\ \tau = \frac{t}{\sqrt{\gamma LC}} \quad (3.6)$$

$$B = \gamma\sqrt{\gamma}\frac{E}{V_s},\quad v = \sqrt{\gamma LC}\,\omega \quad (3.7)$$

and $$\mu\sqrt{\frac{3S}{C}(LS - RC)},\ \gamma = \frac{3LS}{LS-RC}. \quad (3.8)$$

Finally (3.1) will become

$$\frac{dx}{d\tau} = y, \quad (3.9)$$

and $$\frac{dy}{d\tau} = \mu(1 - x^2)y - x^3 + B\ cos(v\tau). \quad (3.10)$$

These systems of the equations describe the forced negative resistance oscillator in the electric circuit. Using a different parameter, we found that it appeared to have periodic oscillations but no chaotic phenomena came from the van der Pol equation with the periodic excitation,

$$\frac{d^2x}{dt^2} - \mu(1 - x^2)\frac{dx}{dt} + x = B\cos(vt), \quad (3.11)$$

when $0 < \mu < 1$.

Ueda and Akamatsu discovered their modified van der Pol equation in their 1981 papers (Ueda & Akamatsu, 1981) which is

$$\frac{d^2x}{dt^2} - (1 - x^2)\frac{dx}{dt} + x^3 = B\cos(\omega t). \quad (3.12)$$

For the convenience sake in this research, we set $\omega = 1$. Here we introduce μ in to the middle term of the left-hand side of an equation where μ is strength of the non-linear parameter. The equation is expressed by

$$\frac{d^2x}{dt^2} - \mu(1 - x^2)\frac{dx}{dt} + x^3 = B\ \cos(t). \quad (3.13)$$

Change (3.13) from second order differential equation to the system of two first order differential equations we will have

$$\frac{dx}{dt} = y, \tag{3.14}$$

$$\frac{dy}{dt} = \mu(1 - x^2)y - x^3 + B\cos(t). \tag{3.15}$$

We wanted to calculate the solutions and study their behaviors. We calculated the phase space, a geometric representation of the dynamical systems. It showed the relationship between domain x and the first derivative of the domain $\frac{dx}{dt}$. The phase space would show if van der Pol equations, have a repellor, an attractor or had limited cycles.

The power spectral density is

$$S_{xx}(\omega) = \lim_{T\to\infty} \boldsymbol{E}\left[|\hat{x}(\omega)|^2\right], \tag{3.16}$$

Where **E** is the expect value and $\hat{x}(\omega)$ is a Fourier transform.

We plotted the graph for $(x, y), (x, \ddot{x}), (t, (x, y)),$ and power spectrum with different parameters $\mu, B, x(0), y(0)$ and the number of steps to study their chaotic behaviors. These results will be discussed later. Special attention was also given to the results whether there was chaos or limit cycles. Finally, we compared the properties between the van der Pol equation and the modified van der Pol equation.

For the force negative-resistance oscillator when we injected the periodic excitation to the system, governed by these systems of the equations (3.14) and (3.15) the results would be either synchronized periodic oscillation or asynchronized nonperiodic oscillation. For the case of asynchronized nonperiodic oscillation it was either chaotic oscillation or almost periodic oscillation. We would investigate the case of almost periodic oscillation, with the different values of parameter B, ν and μ to show the invariant closed curves and average power spectra.

All the results will be reported in chapter 4 and discussed in chapter 5. The Fortran 90 codes used in this research are also shown in the appendix.

CHAPTER 4

RESULTS AND DISCUSSIONS

In this chapter all the results will be discussed. For the modified van der Pol equation the governing equation is

$$\frac{d^2x}{dt^2} - \mu(1 - x^2)\frac{dx}{dt} + x^3 = B\cos(t) \quad (4.1)$$

We changed the parameters and mainly focused on whether the modified van der Pol equation had a limit cycle or nonexistence of a limit cycle. First, we plotted the phase plane in 2 dimensions with different parameters. The phase plane showed the characteristics of the differential equations as follows:

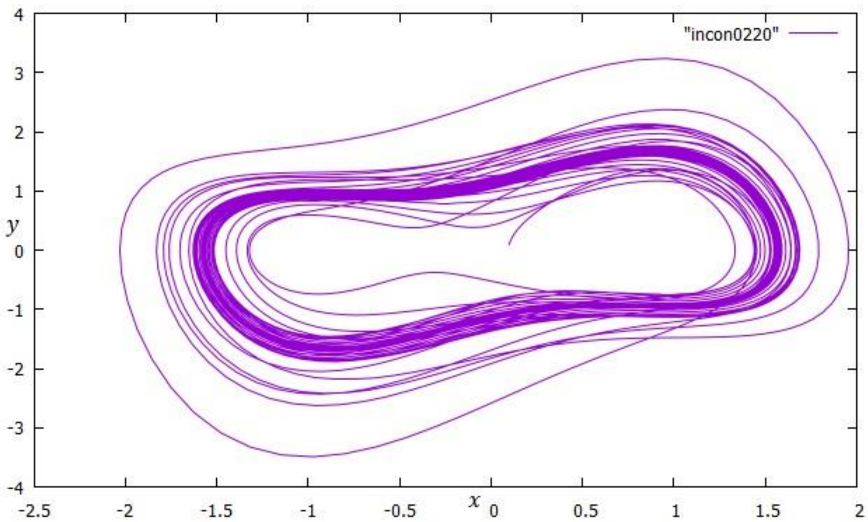

Figure 15 The modified van der Pol equation (x, y) phase plane with $\mu = 1, B = 1$.

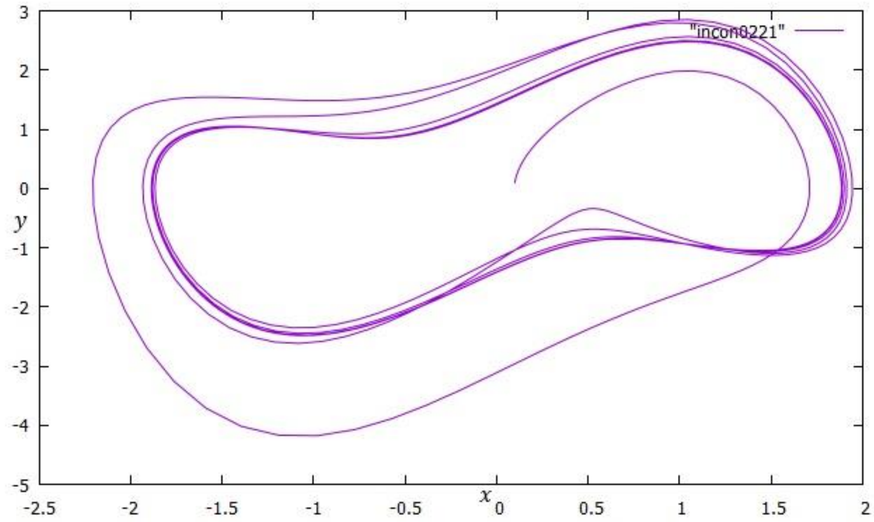

Figure 16 The modified van der Pol equation (x, y) phase plane with $\mu = 1, B = 2$.

Figure 15 shows the modified van der Pol equation (x, y) phase plane with $\mu = 1, B = 1$, initial condition $x(0) = 0.1$ and $y(0) = 0.1$. Figure 16 shows the modified van der Pol equation (x, y) phase plane with $\mu = 1, B = 2$, initial condition $x(0) = 0.1$ and $y(0) = 0.1$.

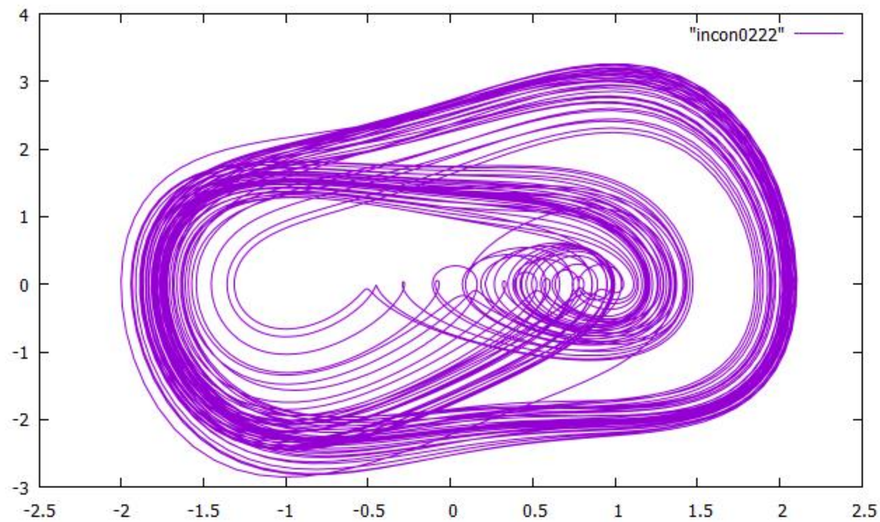

Figure 17 The modified van der Pol equation (x, y) phase plane with $\mu = 0.5, B = 1$.

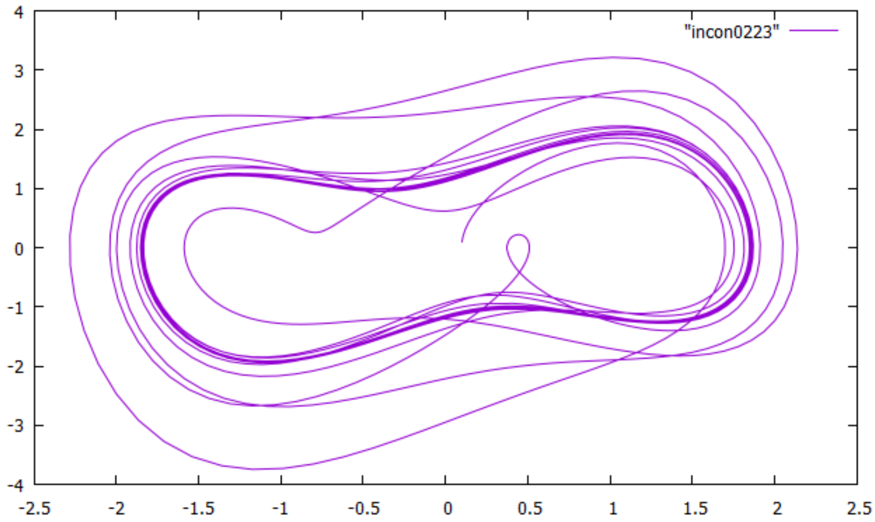

Figure 18 The modified van der Pol equation (x, y) phase plane with $\mu = 0.5, B = 2$.

Figure 17 shows the modified van der Pol equation (x, y) phase plane with $\mu = 0.5, B = 1$, initial condition $x(0) = 0.1$ and $y(0) = 0.1$. Figure 18 shows the modified van der Pol equation (x, y) phase plane with $\mu = 0.5, B = 2$, initial condition $x(0) = 0.1$ and $y(0) = 0.1$.

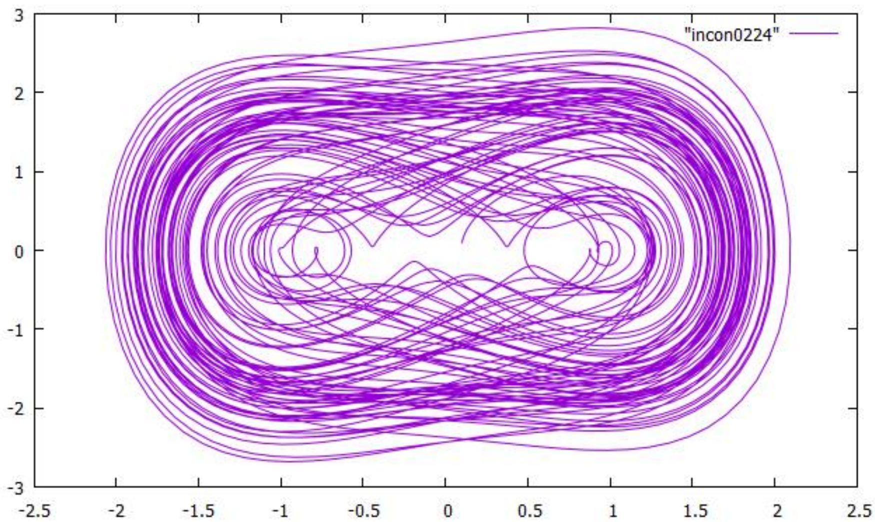

Figure 19 The modified van der Pol equation (x, y) phase plane with $\mu = 0.1, B = 1$.

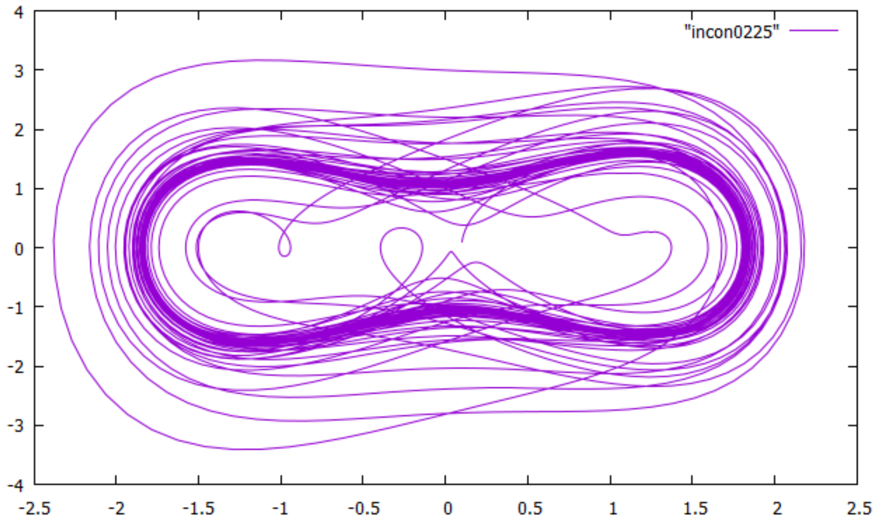

Figure 20 The modified van der Pol equation (x, y) phase plane with $\mu = 0.1, B = 2$.

Figure 19 shows the modified van der Pol equation (x, y) phase plane with $\mu = 0.1, B = 1$, initial condition $x(0) = 0.1$ and $y(0) = 0.1$. Figure 20 shows the modified van der Pol equation (x, y) phase plane with $\mu = 0.1, B = 2$, initial condition $x(0) = 0.1$ and $y(0) = 0.1$.

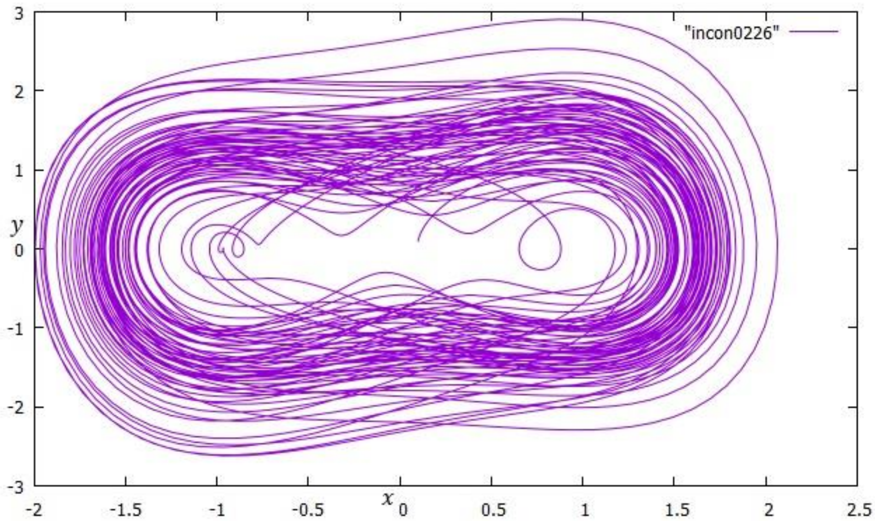

Figure 21 The modified van der Pol equation (x, y) phase plane with $\mu = 0.25, B = 1$.

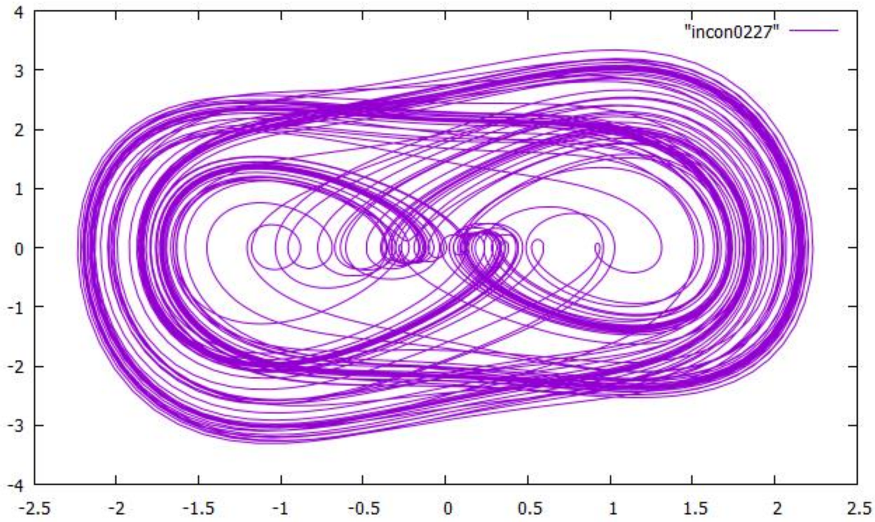

Figure 22 The modified van der Pol equation (x, y) phase plane with $\mu = 0.25, B = 1.5$.

Figure 21 shows the modified van der Pol equation (x, y) phase plane with $\mu = 0.25, B = 1$, initial condition $x(0) = 0.1$ and $y(0) = 0.1$. Figure 22 shows the modified van der Pol equation (x, y) phase plane with $\mu = 0.25, B = 1.5$, initial condition $x(0) = 0.1$ and $y(0) = 0.1$.

Table 4 The modified van der Pol equation (x, y) phase plane with different parameters.

Results	$x(0)$	$y(0)$	Step size (h)	Number of steps (N)	μ	B	Remarks
1	0.1	0.1	0.05	8 000	1	1	Nonexistence of a limit cycle
2	0.1	0.1	0.05	8 000	1	2	Limit cycle
3	0.1	0.1	0.05	8 000	0.5	1	Nonexistence of a limit cycle
4	0.1	0.1	0.05	8 000	0.5	2	Limit cycle
5	0.1	0.1	0.05	8 000	0.1	1	Nonexistence of a limit cycle
6	0.1	0.1	0.05	8 000	0.1	2	Nonexistence of a limit cycle
7	0.1	0.1	0.05	8 000	0.25	1	Nonexistence of a limit cycle
8	0.1	0.1	0.05	8 000	0.25	1.5	Nonexistence of a limit cycle

Figures 15-22 show the phase plane of the modified van der Pol equation. Phase plane shows a plane in which all possible states of the system are represented with each possible state corresponding to one unique point in a phase plane. From table 4 we found that from the total of 8 results, 2 results where $\mu = 1, B = 2$ and $\mu = 0.5, B = 2$ had a limit cycle, the other 6 results had nonexistence of a limit cycle. A limit cycle can be defined as an isolated closed trajectory in 2-dimensional phase plane. It is a periodic motion that arises from a self-excited or an autonomous system such as the systems of electrical oscillation or in aeroelastic flutter. There are 3 kinds of a limit cycle, stable, unstable and semi-stable (Francis, 1987). Limit cycle is very important to analyses of non-linear systems. It is an isolated closed trajectory. For the word "isolated", we mean that the neighboring trajectories are not closed, they spiral either away from a limit cycle or toward a limit cycle. We found that the limit cycle in phase plane depends on 2 parameters, B and μ where B is a co-efficient of a driving force and μ is a strength of a non-linear parameter. We have not found a generalized formula for the case of a limit cycle. From table 4 The results 1 to 5 correspond with Ku and Sun's work (Ku & Sun, 1990). We can conclude that the modified van der Pol equation has a limit cycle and if a driving force is zero, we always get a limit cycle. If a driving force is not zero, we may or may not get a limit cycle.

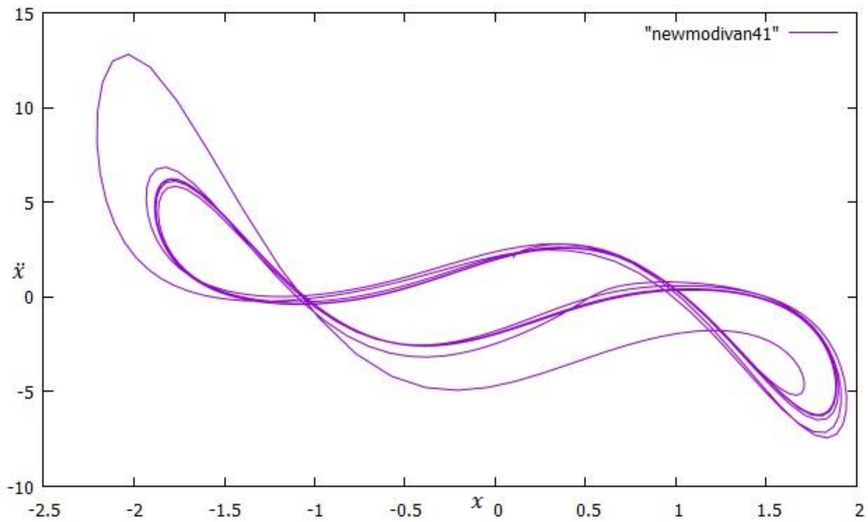

Figure 23 The modified van der Pol equation $(x, \ddot{x})$ graph with $\mu = 1, B = 2$.

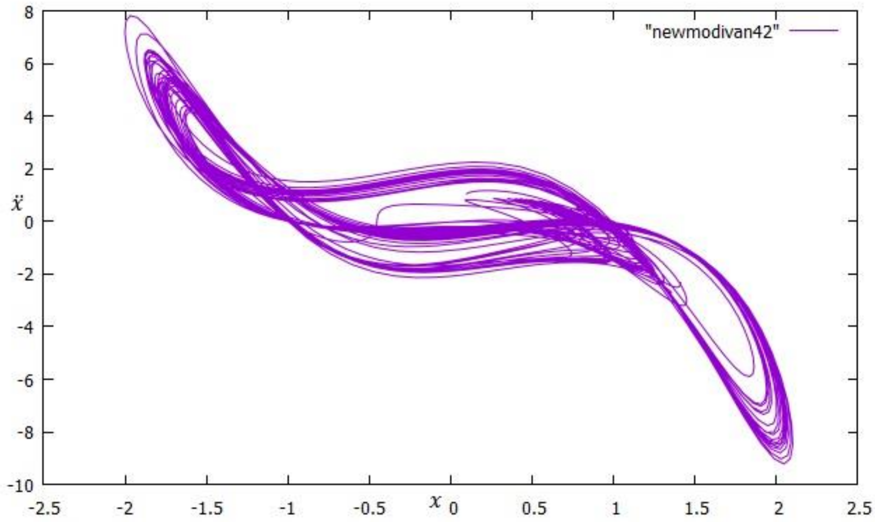

Figure 24 The modified van der Pol equation $(x, \ddot{x})$ graph with $\mu = 0.5, B = 1$.

Figure 23 shows the modified van der Pol equation $(x, \ddot{x})$ graph with $\mu = 1, B = 2$, step size (h) = 0.05, initial condition $x(0) = 0.1$ and $y(0) = 0.1$. Figure 24 shows the modified van der Pol equation $(x, \ddot{x})$ graph with $\mu = 0.5, B = 1$, step size (h) = 0.05, initial condition $x(0) = 0.1$ and $y(0) = 0.1$.

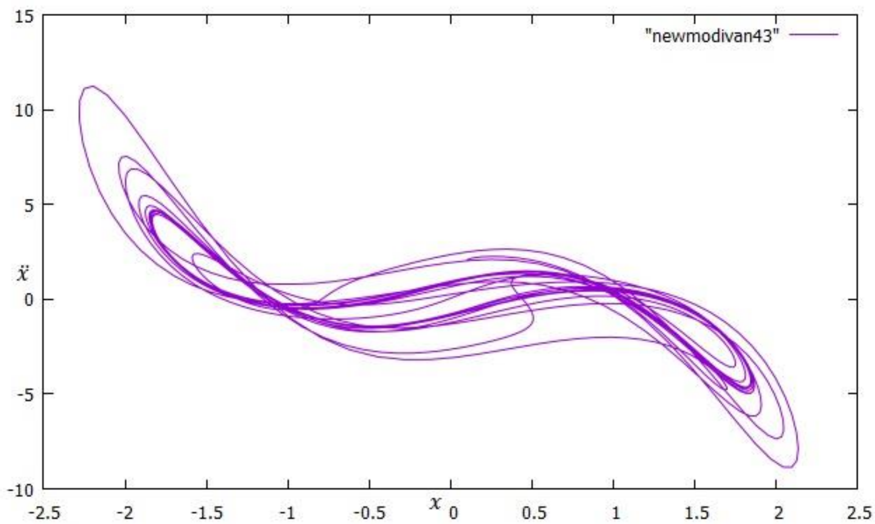

Figure 25 The modified van der Pol equation $(x, \ddot{x})$ graph with $\mu = 0.5, B = 2$.

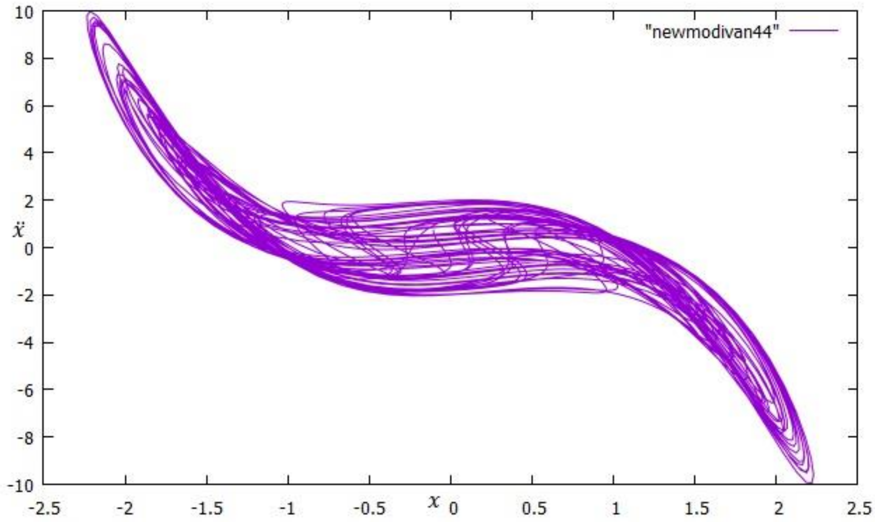

Figure 26 The modified van der Pol equation $(x, \ddot{x})$ graph with $\mu = 0.25, B = 1.5$.

Figure 25 shows the modified van der Pol equation $(x, \ddot{x})$ graph with $\mu = 0.5, B = 2$, step size (h) = 0.05, initial condition $x(0) = 0.1$ and $y(0) = 0.1$. Figure 26 shows the modified van der Pol equation $(x, \ddot{x})$ graph with $\mu = 0.25, B = 1.5$, step size (h) = 0.05, initial condition $x(0) = 0.1$ and $y(0) = 0.1$.

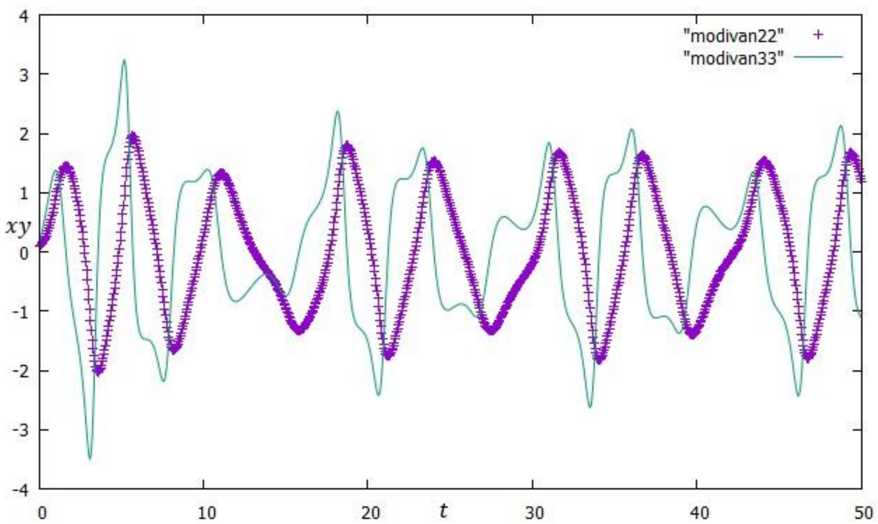

Figure 27 The modified van der Pol equation $(t, (x, y))$ graph with $\mu = 1, B = 2$ (x is violet and y is green).

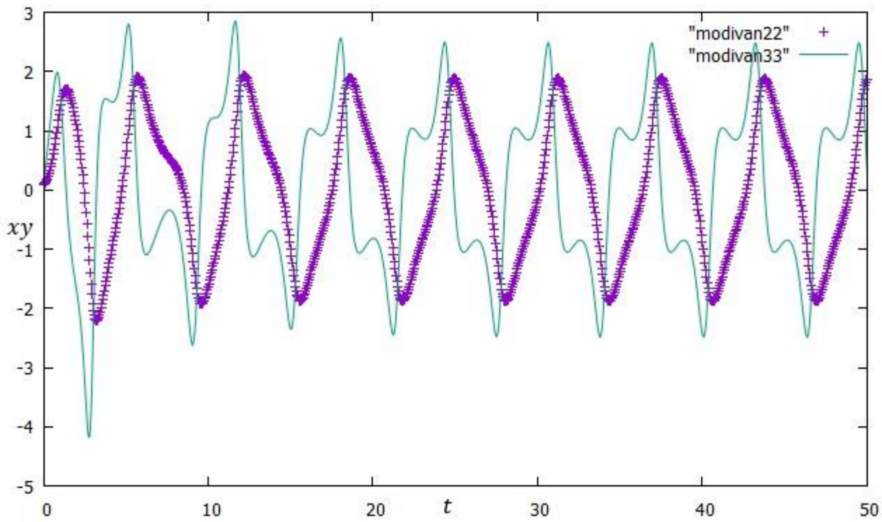

Figure 28 The modified van der Pol equation $(t,(x,y))$ graph with $\mu = 0.5, B = 1$ (x is violet and y is green).

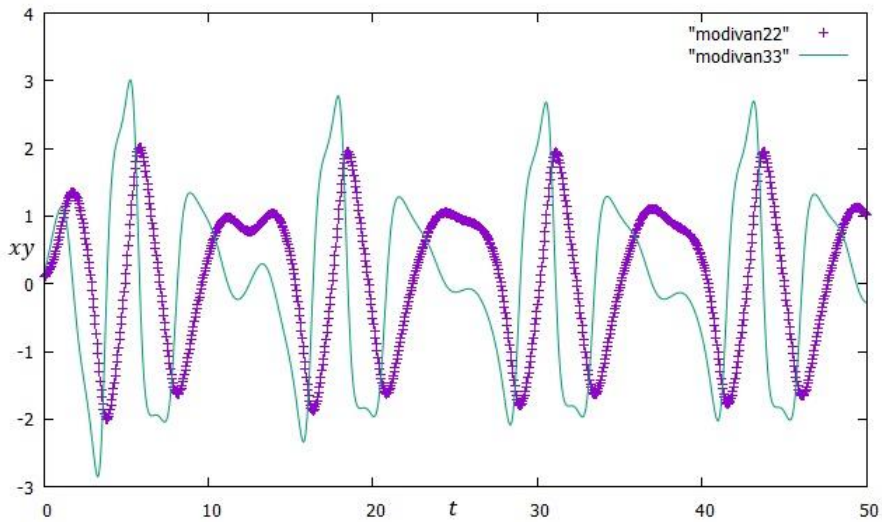

Figure 29 The modified van der Pol equation $(t,(x,y))$ graph with $\mu = 0.5, B = 2$ (x is violet and y is green).

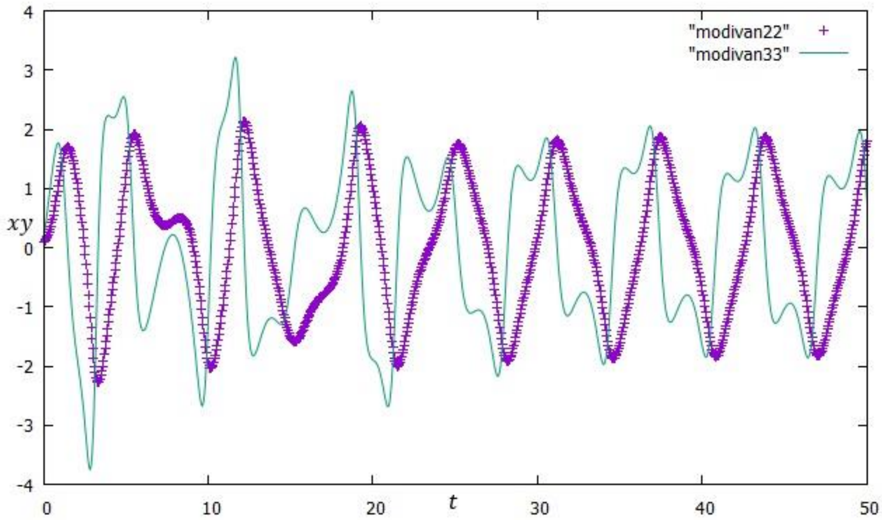

Figure 30 The modified van der Pol equation $(t,(x,y))$ graph with $\mu = 0.25, B = 1.5$ (x is violet and y is green).

Time series is a set of data in time order in successive equally spaced points in time. We used time series to analyze data for differential equations. In Figures 27-30 we used time series to compare between the value of x and y with different parameters. Figure 27 shows the modified van der Pol equation $(t,(x,y))$ graph with $\mu = 1, B = 2$ and step size $(h) = 0.05$. Figure 28 shows the modified van der Pol equation $(t,(x,y))$ graph with $\mu = 0.5, B = 1$ and step size $(h) = 0.05$. Figure 29 shows the modified van der Pol equation $(t,(x,y))$ graph with $\mu = 0.5, B = 2$ and step size $(h) = 0.05$. Figure 30 shows the modified van der Pol equation $(t,(x,y))$ graph with $\mu = 0.25, B = 1.5$ and step size $(h) = 0.05$.

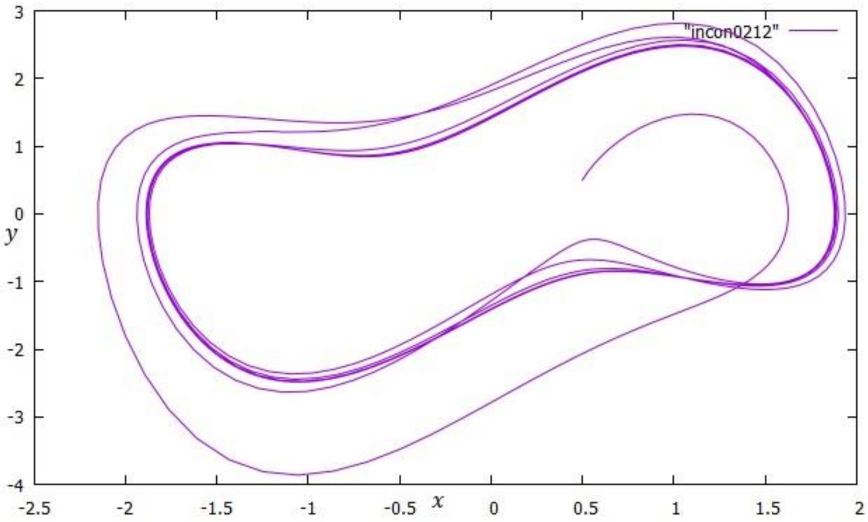

Figure 31 The modified van der Pol equation (x, y) phase plane with initial condition $x(0) = 0.5, y(0) = 0.5$ and $\mu = 1, B = 2$.

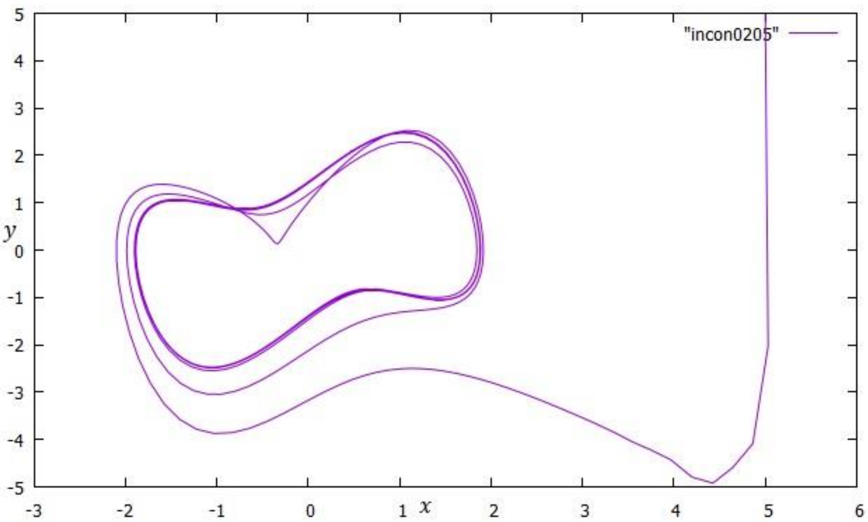

Figure 32 The modified van der Pol equation (x, y) phase plane with initial condition $x(0) = 5, y(0) = 5$ and $\mu = 1, B = 2$.

Figure 31 shows the modified van der Pol equation (x, y) phase plane with initial condition $x(0) = 0.5, y(0) = 0.5, \mu = 1, B = 2$ and number of steps $(N) = 8000$. Figure 32 shows the modified van der Pol equation (x, y)

phase plane with initial condition $x(0) = 5, y(0) = 5, \mu = 1, B = 2$ and number of steps $(N) = 8000$.

The initial condition is a set of starting point values at some point in time. For some non-linear systems initial conditions can affect whether the system converges or diverges when the time t changes. For the case of a limit cycle, figures 33 and 34 show that the behavior does not depend on the initial condition. The initial condition can be at any point but not $(0,0)$ because this is a fixed point. At the point $\dot{x} = 0$ there is no flow. At $(\dot{x}, \dot{y}) = (0,0)$ when $(x, y) = (0,0)$, when time t changes it will remain there forever.

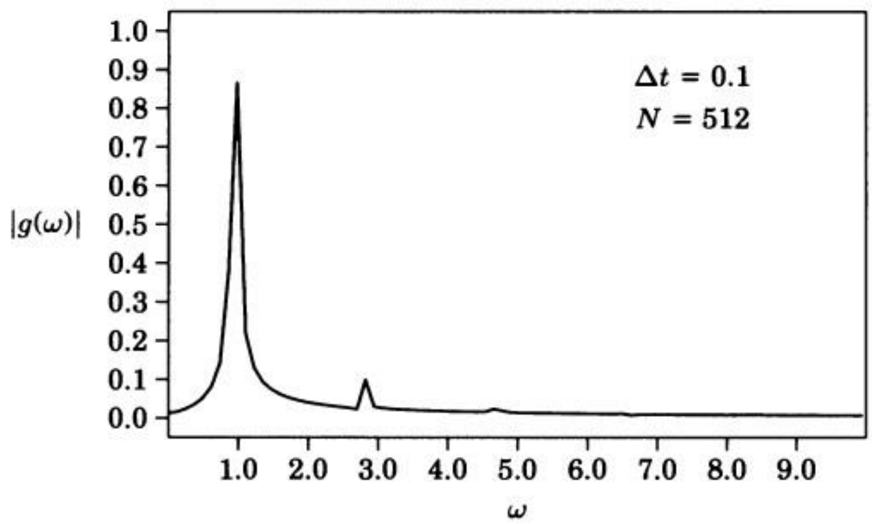

Figure 33 The van der Pol equation power spectral density (PSD) with $\Delta t = 0.1$ and $N = 512$.

Source: Paul L. DeVries. (2011). A first Course in Computational Physics p.322

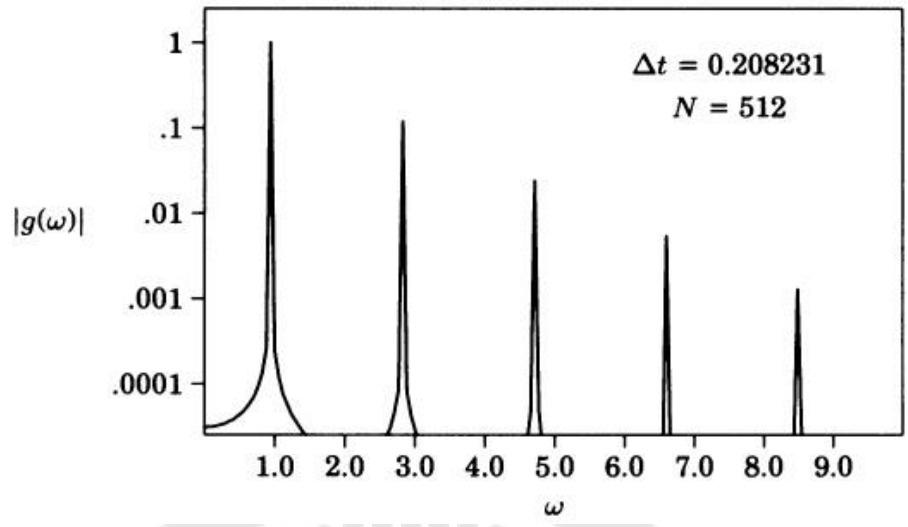

Figure 34 The van der Pol equation power spectral density (PSD) with $\Delta t = 0.208231$ and $N = 512$.

Source: Paul L. DeVries. (2011). A first Course in Computational Physics p.324

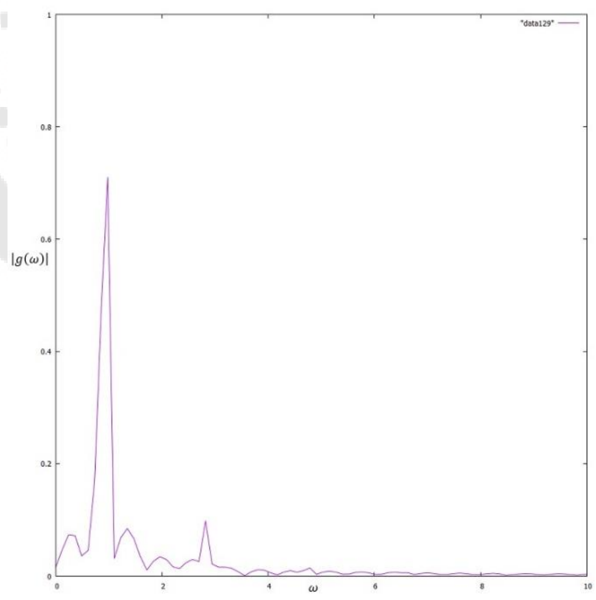

Figure 35 The van der Pol equation power spectral density (PSD) with $\Delta t = 0.1$ and $N = 512$, using gnuplot.

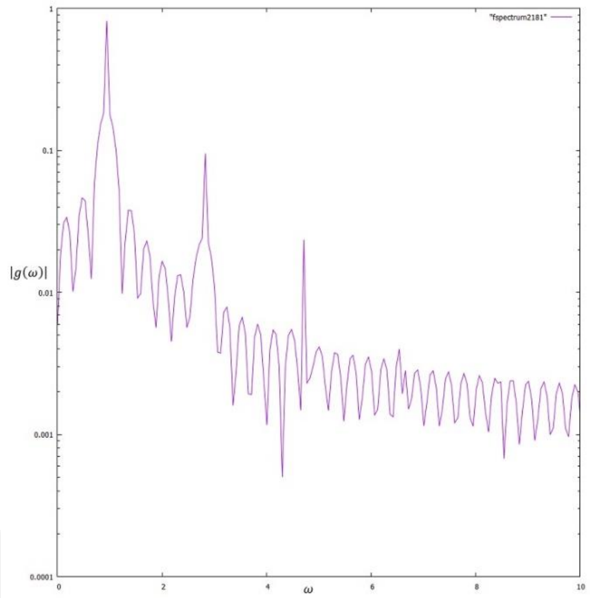

Figure 36 The van der Pol equation power spectral density (PSD) with $\Delta t = 0.208231$ and $N = 512$, using gnuplot.

Figure 33 shows the van der Pol equation spectral density (PSD) with $\Delta t = 0.1$ and $N = 512$. We can see the highest peak at $\omega = 1$. There are other peaks but it is not obvious. We can change from linear scale to a logarithmic scale. Figure 34 shows the van der Pol equation spectral density (PSD) with $\Delta t = 0.208231$ and $N = 512$ using logarithmic scale. By changing Δt from 0.1 to 0.208231 we can get rid of the background noise and can observe other peaks. We compared figure 33-34 with figure 35-36 which came from using gnuplot, and found the results were the same but figure 35-36 had small fluctuations along the curves that may be caused by gnuplot.

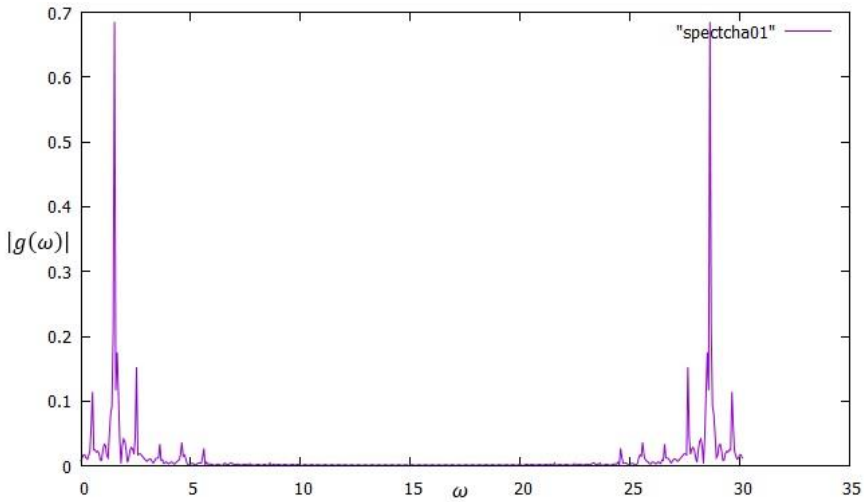

Figure 37 The modified van der Pol equation power spectral density (PSD).

Figure 37 shows the modified van der Pol equation power spectral density (PSD). We used power spectrum because some cases the spectrum of a signal could show aspects of a signal that may not be obvious when looking at a time-domain representation only. Power spectrum shows the strength of the variations as a function of frequency which means it shows which frequency is strong and which frequency is weak. We can compute power spectrum by a method called Fast Fourier Transform (FFT), an algorithm that computes the discrete Fourier transform of a sequence. Figure 37 shows power spectral density has the high peak at $\omega = 2$ and 28 , the low peak at $\omega = 1,3,27$ and 29.

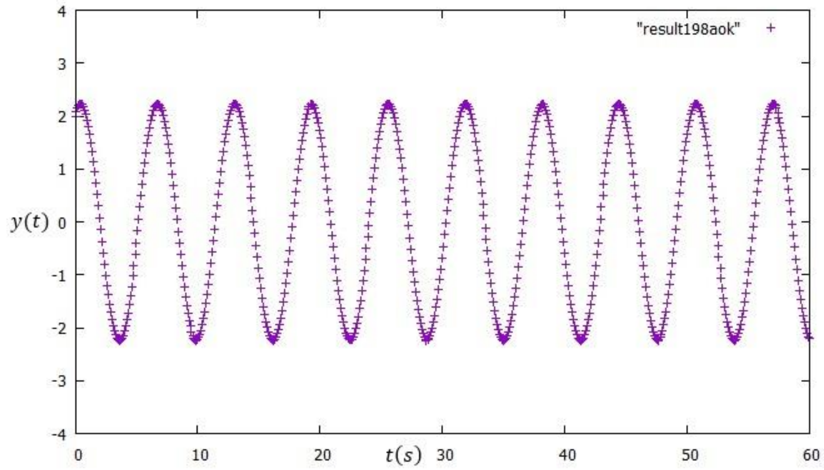

Figure 38 The van der Pol equation plot of $y(t)$ against $t(s)$ for $\mu = 0$.

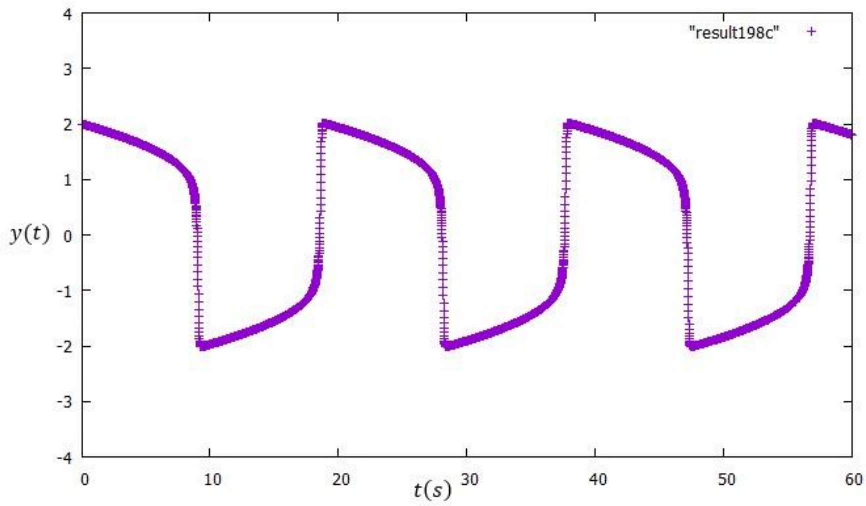

Figure 39 The van der Pol equation plot of $y(t)$ against $t(s)$ for $\mu = 10$.

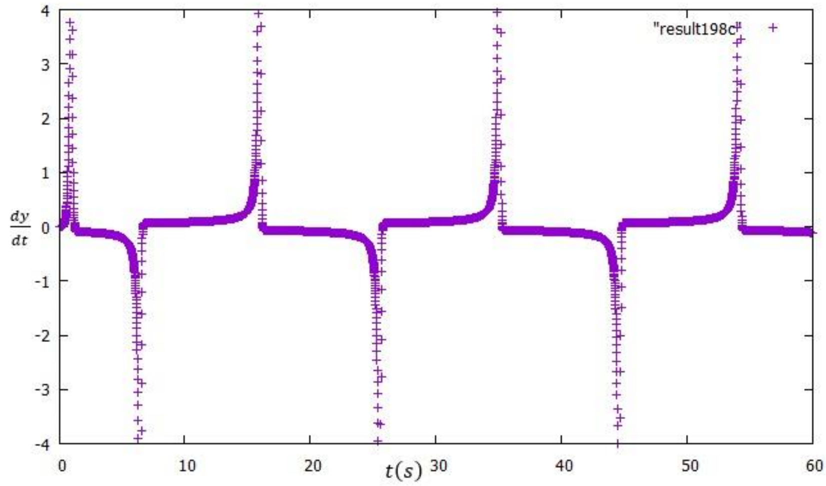

Figure 40 The van der Pol equation plot of $\frac{dy}{dt}$ against $t(s)$ for $\mu = 10$.

We compared the van der Pol equation and the modified van der Pol equation, which has the term x^3 instead of x and also had the driving force $B\cos(t)$. The van der Pol equation has a unique and stable limit cycle. From table 4 we found that the modified van der Pol equation sometimes had a limit cycle. In both cases μ was a strength parameter of a non-linear term. For the van der Pol equation if $\mu = 0$, the equation will become a simple harmonic oscillator. Figures 39 and 40 show the van der Pol equation plot of $y(t)$ and $\frac{dy}{dt}$ respectively against $t(s)$ for $\mu = 10$. Figure 38 shows the plot of $y(t)$ against $t(s)$ from $t = 0 - 60$, for $\mu = 0$. The results correspond with Hafeez's work (Hafeez et al., 2015).

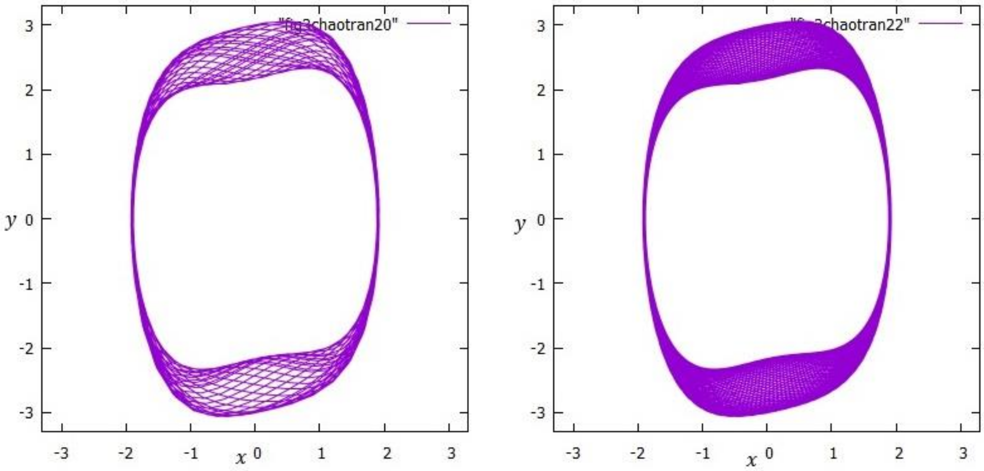

Figure 41 The modified van der Pol equation (x, y) phase plane in the case of almost periodic oscillation (left figure has the number of steps $= 10\,000$, right figure has the number of steps $= 50\,000$).

For the force negative-resistance oscillator which was governed by the modified van der Pol equation. The results would be either synchronized periodic oscillation or asynchronized nonperiodic oscillation. For the case of asynchronized nonperiodic oscillation it was either chaotic oscillation or almost periodic oscillation. Figure 41 shows the modified van der Pol equation (x, y) phase plane in the case of almost periodic oscillation. Almost periodic oscillation can be defined as a continuous close figure or in orbit, sometimes it is called “motion on a torus” (Francis, 1987). It is a property of the dynamical system that in a 2-dimensional phase plane it appears to retrace its trajectory, but not exactly. Almost periodic means that it is periodic within any desired level of accuracy given in a long period. We found that in this case when the parameter $\mu = 0.2, \nu = 4$ and $B = 1$, the phase space would have the invariant closed curve which was different from chaotic oscillation.

In this research we used Runge–Kutta methods (RK4) , and the Adams-Bashforth-Moulton methods (ABM) for numerical analysis. Comparing these methods to Euler’s method and Heun’s method, these methods give a more accurate approximation.

We also used the Liénard theorem to analyze the systems. The van der Pol equation which has $f(x) = \mu(x^2 - 1)$ and $g(x) = x$, will satisfy all the conditions of the Liénard theorem (1.) $f(x)$ and $g(x)$ are continuously differentiable for all values of x. (2.) $g(-x) = -g(x)$ (3.) $g(x) > 0$ for $x > 0$. (4.) $f(-x) = f(x)$ and (5.) The odd function $F(x) = \int_0^x f(u)du$ has exactly one positive zero at $x = a,$ is negative for $0 < x < a,$ is positive and nondecreasing for $x > a,$ and $F(x) \to \infty$ as $x \to \infty$.

The modified van der Pol equation which has $f(x) = \mu(x^2 - 1)$ and $g(x) = x^3$ satisfied conditions (1.)-(4.) but It is not a Liénard equation of the form $\ddot{x} + f(x)\dot{x} + g(x) = 0$ because it has a driving force term $B\cos(t)$.Table 4 shows that the modified van der Pol equation may have a limit cycle, which means that the Liénard theorem is sufficient but not necessary for a limit cycle. For the theory of averaging for the modified van der Pol equation, the behavior also depends on the parameters.

CHAPTER 5
CONCLUSIONS

In this chapter, we will briefly conclude literature reviews, methods, results and discussions. Balthasar van der Pol (1889-1959) founded a governing equation that was named after him in electrical circuits in 1926. Relaxation-oscilation is a property in the van der Pol equation. In 1927 van der Pol introduced a forced van der Pol equation. After that, many researches have been done in various fields about van der Pol equation. In 1981 Ueda and Akamatsu discovered the modified van der Pol equation in the forced negative-resistance oscillator.

There are many numerical methods for solving differential equations such as the Euler method, the Heun's method, the Runge-kutta methods, the Runge-Kutta-Fehlberg method and the Adam-Bashforth_Moulton methods. In this research we study the non-linearity in the circuit systems. We use the Runge–Kutta methods (RK4), and the Adams-Bashforth-Moulton methods (ABM) to study the modified van der Pol equation with different parameters.

We plotted the graph for $(x, y), (\ddot{x}, y), ((x, y), t)$, and the power spectrum. We found from 2-dimensional phase plane that the modified van der Pol equation had a limit cycle at where $\mu = 1, B = 2$ and $\mu = 0.5, B = 2$ from the total of 8 results. A limit cycle depends on 2 parameters B and μ. The initial condition is a set of starting point values at some point in time. For the case of a limit cycle the behavior does not depend on the initial condition. Power spectrum shows the strength of the variation as a function of frequency. By using the fast Fourier transform we found the modified van der Pol equation has power spectral density in high peak at $\omega = 2$ and 28, and small peak at $\omega = 1,3,27$ and 29

We compared the van der Pol equation and the modified van der Pol equation, which has the term x^3 instead of x and also has the driving force $B\cos(t)$. In the case of almost periodic oscillation we found that when the parameter was $\mu =$

$0.2, \nu = 4$ and $B = 1$ the phase space would have the invariant closed curve which was different from chaotic oscillation.

APPENDIX

Examples of Fortran90 Programming Language Code

1 2D Phase plane

```
implicit real*8 (a-h,o-z)
real*8 x,y,h,fx0,fy0
real*8 mu,amp,w
dimension t(0:50000),y(0:50000),x(0:50000)
common mu,amp,w,bigf,omega,epsilon
write(6,'("enter value of bigf")')
read(5,*)bigf
write(6,'("enter value of omega")')
read(5,*)omega
write(6,'("enter value of epsilon")')
read(5,*)epsilon
write(6,'("enter value of  x(0) and y(0)  ")')
 read(5,*)x(0),y(0)
  write(6,'("enter value of  time increment and number of steps  ")')
read(5,*)h,n

a=0.d0
b=n*h
     open(8,file='incon02')
      call abm(a,b,n,t,x,y)
      do i=0,n
        write (8,'(2e18.8)')x(i),y(i)
       end do
  end

subroutine abm(a,b,n,t,x,y)
implicit real*8 (a-h,o-z)
dimension t(0:*),y(0:*),x(0:*)
```

```
      h=(b-a)/dfloat(n)
      t(0)=a
      fx0=fx(t(0),x(0),y(0))
      fy0=fy(t(0),x(0),y(0))

      call rk4(t,x,y,h,3)
      do 10 k=1,3
10    continue

      fx1= fx(t(1),x(1),y(1))
      fy1= fy(t(1),x(1),y(1))
      fx2= fx(t(2),x(2),y(2))
      fy2= fy(t(2),x(2),y(2))
      fx3= fx(t(3),x(3),y(3))
      fy3= fy(t(3),x(3),y(3))
      h2=h/24.d0

      write (6,'("a,b",2f12.6)')a,b

      do 20 k=3, (n-1)
      px=x(k)+h2*(-9.d0*fx0+37.d0*fx1-59.d0*fx2+55.d0*fx3)
      py=y(k)+h2*(-9.d0*fy0+37.d0*fy1-59.d0*fy2+55.d0*fy3)
      t(k+1)=a+h*(k+1)
      fx4=fx(t(k+1),px,py)
      fy4=fy(t(k+1),px,py)
      x(k+1)=x(k)+h2*(fx1-5.d0*fx2+19.d0*fx3+9.d0*fx4)
      y(k+1)=y(k)+h2*(fy1-5.d0*fy2+19.d0*fy3+9.d0*fy4)
      fx0=fx1
      fy0=fy1
```

```
      fx1=fx2
      fy1=fy2
      fx2=fx3
      fy2=fy3
      fx3=fx(t(k+1),x(k+1),y(k+1))
      fy3=fy(t(k+1),x(k+1),y(k+1))
 20   continue

      return
      end

      subroutine rk4 (t,x,y,h,m)
      implicit real*8 (a-h,o-z)
      dimension t(0:100),y(0:100),x(0:100)
      do 10 j=0,(m-1)
      fx1=h*fx(t(j),x(j),y(j))
      fy1=h*fy(t(j),x(j),y(j))
      fx2=h*fx(t(j)+.5d0*h,x(j)+.5d0*fx1,y(j)+.5d0*fy1)
      fy2=h*fy(t(j)+.5d0*h,x(j)+.5d0*fx1,y(j)+.5d0*fy1)
      fx3=h*fx(t(j)+.5d0*h,x(j)+.5d0*fx2,y(j)+.5d0*fy2)
      fy3=h*fy(t(j)+.5d0*h,x(j)+.5d0*fx2,y(j)+.5d0*fy2)
      fx4=h*fx(t(j)+h,x(j)+fx3,y(j)+fy3)
      fy4=h*fy(t(j)+h,x(j)+fx3,y(j)+fy3)
      x(j+1)=x(j)+(fx1+2.d0*fx2+2.d0*fx3+fx4)/6.d0
      y(j+1)=y(j)+(fy1+2.d0*fy2+2.d0*fy3+fy4)/6.d0
      t(j+1)=t(j)+h
 10   continue
      return
      end
```

```
function fy(t,x,y)
real*8 fy,t,y,x ,mu ,amp,w ,bigf,epsilon,omega
 common mu,amp,w,bigf,omega,epsilon
   fy=(epsilon)*(1.d0-(x*x))*y-(x*x*x) +(bigf*cos(t))
return
end

function fx(t,x,y)
real*8 fx,t,y,x ,mu
 common mu
  fx=y
return
end
```

2 Power spectrum

```
double precision function fx(t,x,y)
double precision t,x,y

  fx=y
  end

double precision function fy(t,x,y)
double precision t,x,y,eps
common /blockfy/ eps
  fy=-x-eps*(x*x-1.d0)*y
end

implicit none
```

```
double precision fx,fy
external fx,fy
double precision a2,a3,a4,a5,a6,     &
              b21,b31,b32,b41,b42,b43,b51,b52,b53,b54,     &
               b61,b62,b63,b64,b65, &
               w1,w2,w3,w4,w5,w6, &
               e1,e2,e3,e4,e5,e6
double precision h0,x0,y0,h,tol,tmax,eps
double precision t,x,y
common /blockfy/ eps
double precision f11,f12,f21,f22,f31,f32,f41,f42,f51,f52,f61,f62
double precision errest
double precision dt,next,pi,dw
integer i,n,nn,j,m,k,istep
complex*16 f(0:1023),z,w,wp,temp

a2=1.d0/4.d0
a3=3.d0/8.d0
a4=12.d0/13.d0
a5=1.d0
a6=1.d0/2.d0

b21=1.d0/4.d0
b31=3.d0/32.d0
b32=9.d0/32.d0
b41=1932.d0/2197.d0
b42=-7200.d0/2197.d0
b43=7296.d0/2197.d0
b51=439.d0/216.d0
```

```
b52=-8.d0
b53=3680.d0/513.d0
b54=-845.d0/4104.d0
b61=-8.d0/27.d0
b62=2.d0
b63=-3544.d0/2565.d0
b64=1859.d0/4104.d0
b65=-11.d0/40.d0

w1=25.d0/216.d0
w2=0.d0
w3=1408.d0/2565.d0
w4=2197.d0/4104.d0
w5=-1.d0/5.d0
w6=0.d0

e1=1.d0/360.d0
e2=0.d0
e3=-128.d0/4275.d0
e4=-2197.d0/75240.d0
e5=1.d0/50.d0
e6=2.d0/55.d0
x0=0.5d0
y0=0.0d0
eps=1.d0
dt=0.208231d0
n=512+100
h0=0.001d0
tol=1.e-6
```

```
x=x0
y=y0

open (1,file="spectrum101.dat")

t=0.d0
do i=1,n
next=i*dt
h=h0

1 continue

if (t+h.gt.next) h=next-t
if (h.le.0.d0)then
  t=next
  if(next.ge.100*dt)then
    f(i-100)=x
    write (1,'(2e24.16)')next-100*dt,x
  endif
  cycle
endif

2 continue

 f11=h*fx(t,x,y)
 f12=h*fy(t,x,y)
 f21=h*fx(t+a2*h,x+b21*f11,y+b21*f12)
 f22=h*fy(t+a2*h,x+b21*f11,y+b21*f12)
 f31=h*fx(t+a3*h,x+b31*f11+b32*f21,y+b31*f12+b32*f22)
```

```
 f32=h*fy(t+a3*h,x+b31*f11+b32*f21,y+b31*f12+b32*f22)
 f41=h*fx(t+a4*h,x+b41*f11+b42*f21+b43*f31,  &
               y+b41*f12+b42*f22+b43*f32)
 f42=h*fy(t+a4*h,x+b41*f11+b42*f21+b43*f31,    &
               y+b41*f12+b42*f22+b43*f32)
 f51=h*fx(t+a5*h,x+b51*f11+b52*f21+b53*f31+b54*f41,  &
           y+b51*f12+b52*f22+b53*f32+b54*f42)
 f52=h*fy(t+a5*h,x+b51*f11+b52*f21+b53*f31+b54*f41,  &
           y+b51*f12+b52*f22+b53*f32+b54*f42)
 f61=h*fx(t+a6*h,x+b61*f11+b62*f21+b63*f31+b64*f41+b65*f51, &
           y+b61*f12+b62*f22+b63*f32+b64*f42+b65*f52)
 f62=h*fy(t+a6*h,x+b61*f11+b62*f21+b63*f31+b64*f41+b65*f51,  &
           y+b61*f12+b62*f22+b63*f32+b64*f42+b65*f52)
 errest=abs(e1*f11+e2*f21+e3*f31+e4*f41+e5*f51+e6*f61)       &
    +abs(e1*f12+e2*f22+e3*f32+e4*f42+e5*f52+e6*f62)

if (errest.gt.tol*h)then
 h=.5d0*h
goto 2
else
 x=x+w1*f11+w2*f21+w3*f31+w4*f41+w5*f51+w6*f61
 y=y+w1*f12+w2*f22+w3*f32+w4*f42+w5*f52+w6*f62
 t=t+h
 if(errest.lt.tol*h/32.d0)h=2.d0*h
 goto 1
endif
enddo
close(1)
n=512
```

```
nn=n-1
j=0
do 3 i=0,nn
     if (j.gt.i)then
          temp =f(j)
          f(j) =f(i)
          f(i)=temp
     endif
     m=n/2
20     if((m.ge.1).and.(j.ge.m))then
          j=j-m
          m=m/2
          goto 20
       endif
3      j=j+m
     pi=4.d0*datan(1.d0)
     z=(0.d0,1.d0)
     k=1
4    if(n.gt.k)then
       istep=2*k
       wp=exp(z*pi/k)
       w=1.d0
       do 13 m=0,k-1
         do 12 i=m,nn,istep
            j=i+k
            temp=w*f(j)
            f(j)=f(i)-temp
12            f(i)=f(i)+temp
13         w=w*wp
```

```
  k=istep
  goto 4
 end if
 dw=2.d0*pi/(n*dt)
   open (1,file="fspectrum101.dat")
 do 40 i=0,nn
40   write(1,'(2e24.16)')i*dw,abs(f(i))/n
 close(1)
 end
```

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