ON THE DECOMPOSITION OF COMPLETE LEIBNIZ ALGEBRA

SUTIDA PATLERTSIN

Graduate School Srinakharinwirot University

2023

การแยกของพีชคณิตไลบ์นิทซ์แบบบริบูรณ์

สุธิดา พัฒเลิศสินธุ์

ปริญญานิพนธ์นี้เป็นส่วนหนึ่งของการศึกษาตามหลักสูตร
ปรัชญาดุษฎีบัณฑิต สาขาวิชาคณิตศาสตร์
คณะวิทยาศาสตร์ มหาวิทยาลัยศรีนครินทรวิโรฒ
ปีการศึกษา 2566
ลิขสิทธิ์ของมหาวิทยาลัยศรีนครินทรวิโรฒ

ON THE DECOMPOSITION OF COMPLETE LEIBNIZ ALGEBRA

SUTIDA PATLERTSIN

A Dissertation Submitted in Partial Fulfillment of the Requirements

for the Degree of DOCTOR OF PHILOSOPHY

(Mathematics)

Faculty of Science, Srinakharinwirot University

2023

Copyright of Srinakharinwirot University

THE DISSERTATION TITLED

ON THE DECOMPOSITION OF COMPLETE LEIBNIZ ALGEBRA

BY

SUTIDA PATLERTSIN

HAS BEEN APPROVED BY THE GRADUATE SCHOOL IN PARTIAL FULFILLMENT

OF THE REQUIREMENTS FOR THE DOCTOR OF PHILOSOPHY

IN MATHEMATICS AT SRINAKHARINWIROT UNIVERSITY

(Assoc. Prof. Dr. Chatchai Ekpanyaskul, MD.)

Dean of Graduate School

ORAL DEFENSE COMMITTEE

.... Major-advisor

(Asst. Prof.Suchada Pongprasert)

.... Chair

(Assoc. Prof.Keng Wiboonton)

.... Committee

(Assoc. Prof.Varanoot Khemmani)

.... Committee

(LecturerSermsri Thaithae)

.... Committee

(Asst. Prof.Jittinart Rattanamoong)

Title ON THE DECOMPOSITION OF COMPLETE LEIBNIZ ALGEBRA
Author SUTIDA PATLERTSIN
Degree DOCTOR OF PHILOSOPHY
Academic Year 2023
Thesis Advisor Assistant Professor Suchada Pongprasert

Leibniz algebras, generalizations of Lie algebras, are characterized by their non-antisymmetric properties. In this study, we delve into the properties of decompositions within Leibniz algebras, drawing parallels with analogous results in Lie algebras. Our investigation extends to complete Leibniz algebras, focusing on the conditions governing their extensions. Similar to Lie algebras, we find that inner derivations play a pivotal role in characterizing complete Leibniz algebras. Specifically, it was revealed that the algebra of inner derivations of a Leibniz algebra can be decomposed into the sum of the algebra of left multiplications and a certain ideal. Moreover, the quotient of the algebra of derivations of the Leibniz algebra by this ideal yields a complete Lie algebra. The results further demonstrated that any derivation of a semisimple Leibniz algebra can be expressed as a combination of three derivations. Additionally, the properties of the algebra of inner derivations were explored in comparison to the algebra of central derivations. We also delve into the study of generalizations of derivations of Leibniz algebras.

Keyword : Leibniz algebra, Lie algebra, Decomposition, Central derivation, Inner derivation

ACKNOWLEDGEMENTS

I would like to express my gratitude to the following individuals and institutions:

Firstly, I am immensely thankful to Srinakharinwirot University for providing me with a scholarship and the opportunity to pursue further studies. The financial support I received played a crucial role in allowing me to focus on my academic endeavors.

I owe a great deal of appreciation to my advisor, Dr. Suchada Pongprasert, whose guidance, advice, and inspiration were instrumental throughout this study. Dr. Pongprasert's mentorship not only made research in this field enjoyable but also ensured that my work adhered to the highest standards of excellence.

I am also grateful to my committee member and Dr. Thitarie Rungratgasame, for their valuable feedback and insightful advice. Dr. Rungratgasame's input provided me with a fresh perspective on the topic and greatly contributed to the refinement of my work.

Furthermore, I extend my thanks to the Mathematics Department at North Carolina State University, particularly Dr. Kailash C. Misra, for affording me the opportunity to present my thesis abroad in the U.S. I am appreciative of Dr. Misra's time and prompt assistance in addressing any queries I had.

Last but certainly not least, I am deeply indebted to my family. Their unwavering support and encouragement have been the cornerstone of my journey. I am incredibly fortunate to have such a loving and supportive family, and I am proud to be a member of it.

SUTIDA PATLERTSIN

TABLE OF CONTENTS

CHAPTER 1
INTRODUCTION

Lie algebras, introduced by Marius Sophus Lie in the 1870s, serve as fundamental mathematical structures for examining infinitesimal transformations. Lie theory permeates various mathematical disciplines, including harmonic analysis, algebraic topology, algebraic geometry, combinatorics, number theory, and physics (see, for example, (1), (2), (3), (4)). In 1989, Loday (5) noticed that the Chevalley-Eilenberg boundary map on the exterior can be lifted to the tensor algebra of a Lie algebra and introduced a finite dimensional algebra $\mathbf{A}$ over an algebraically closed field $\mathbb{F}$ with a bilinear bracket to be a Leibniz algebra if it satisfies the Leibniz identity $[a,[b,c]] = [[a,b],c] + [b,[a,c]]$ for all $a,b,c \in \mathbf{A}$. Notably, a Leibniz algebra A aligns with a Lie algebra if and only if $[a,a] = 0$ for every element $a \in \mathbf{A}$. Given that Leibniz algebras extend Lie algebras, understanding their properties has become a focal point of research endeavors.

Similar to Lie algebras, derivations play a pivotal role in comprehending the structure and properties of Leibniz algebras. A linear transformation δ: $\mathbf{A} \to \mathbf{A}$ is called a derivation of $\mathbf{A}$ if $\delta[x,y] = [\delta(x),y] + [x, \delta(y)]$ for all $x, y \in \mathbf{A}$. The set of all derivations of $\mathbf{A}$ is denoted by Der($\mathbf{A}$). Notably, Meng (6) established in 1994 that if a Lie algebra $\mathbf{L} = \mathbf{L}_1 \oplus \mathbf{L}_2$, where $\mathbf{L}_1$ and $\mathbf{L}_2$ are ideals of $\mathbf{L}$, then $\mathrm{Der}(\mathbf{L}) = \mathrm{Der}(\mathbf{L}_1) \oplus \mathrm{Der}(\mathbf{L}_2)$. In Lie theory, specific types of Lie algebras, such as complete, nilpotent, simple, and semisimple Lie algebras, garner significant attention. A Lie algebra $\mathbf{L}$ is complete if its center is trivial and all derivations of $\mathbf{L}$ are inner. A Lie algebra $\mathbf{L}$ is nilpotent if $\mathbf{L}^m = \{0\}$ for some positive integer m where $\mathbf{L} = \mathbf{L}^1$, $\mathbf{L}^i = [\mathbf{L}, \mathbf{L}^{i-1}]$ for $i \geqslant 2$. $\mathbf{L}$ is a simple Lie algebra if $\mathbf{L}$ is non-abelian and contains no non-zero proper ideals, and $\mathbf{L}$ is semisimple if it is a direct sum of simple Lie algebras. Jacobson (7) proved in 1979 that all non-zero nilpotent Lie algebras are not complete. However, in 1994, Meng (6) demonstrated that all semisimple Lie algebras are complete. He also showed that a Lie algebra $\mathbf{L}$ is complete if and only if the holomorph of $\mathbf{L}$, $\mathrm{hol}(\mathbf{L}) = \mathbf{L} \oplus \mathrm{Der}(\mathbf{L})$, is a direct sum of $\mathbf{L}$ and the centralizer of $\mathbf{L}$ in the holomorph, i.e., $\mathrm{hol}(\mathbf{L}) = \mathbf{L} \oplus Z_{\mathrm{hol}(\mathbf{L})}(\mathbf{L})$.

The notion of complete Leibniz algebras was introduced in 2013 by Ancochea and Campoamor (8), with a definition identical to that of complete Lie algebras. However, Boyle, Misra, and Stitzinger (9) later refined this concept, introducing a different definition and showcasing a semisimple Leibniz algebra that did not adhere to the previous definition's completeness criterion. They instead defined a complete Leibniz algebra A as one in which the center of A / Leib(A) is trivial, and for every derivation δ of A, there exists $x \in$ A such that im($\delta - L_x$) $\subseteq$ Leib(A). Utilizing this new definition, some fundamental results from Lie theory carry over to Leibniz algebras. Specifically, it has been proven that all nilpotent Leibniz algebras are not complete, and all semisimple Leibniz algebras are complete. However, if A is a complete non-Lie Leibniz algebra, the holomorph of A is not the direct sum of A and $Z_{\text{hol(A)}}$(A). Based on these findings, we will explore complete Leibniz algebras under the definition introduced by Boyle, Misra, and Stitzinger.

This report consists of five chapters. In chapter 2, we review important notions and results of Leibniz algebras. In chapter 3, we focus on the properties of derivations and ideals of Leibniz algebras. We define set I as the set of all derivations of a Leibniz algebra A whose image is a subset of Leib(A) and show that I is a characteristic ideal of A. We also prove that the algebra of inner derivations of a Leibniz algebra can be decomposed into the sum of the algebra of left multiplications and the ideal I. Then, we assume that the Leibniz algebra A is the direct sum of two ideals and study the properties of the decompositions of Leibniz algebras. We demonstrate that the algebra of derivations of a Leibniz algebra cannot be decomposed in the same manner as the algebra of derivations of a Lie algebra. We also provide an example to illustrate this point. Additionally, we study the properties of inner derivations of Leibniz algebras by comparing them with the set of central derivations, as done for Lie algebras by Tôgô in (10). In chapter 4, we prove that the direct sum of complete Leibniz algebras is also complete and any derivation of a semisimple Leibniz algebra can be written as a combination of three derivations in a different approach from Rakhimov, Masutova and Omirov (11). In (6), Meng showed that the Lie algebra of derivations of any complete Lie algebra is complete.

However, in (12) Kongsomprach, Pongprasert, Rungratgasame and Tiansa-ard showed that this result does not hold for some complete Leibniz algebras. We focus on Leibniz algebras with complete liezation and prove that the quotient of the Lie algebra of derivations of these Leibniz algebras by the ideal I is complete. This quotient algebra is isomorphic to the Lie algebra of derivations of the liezation. In the last chapter, we study some properties of generalized derivations of finite dimensional Lie algebras and investigate some analogues of those properties for Leibniz algebras. Throughout this work, all algebras are assumed to be finite dimensional over an algebraically closed field $\mathbb{F}$ with characteristic zero.

CHAPTER 2
PRELIMINARIES

In this chapter, we review definitions and facts that will be needed later in our discussion.

Definition 2.1. (13) A *Lie algebra* **L** is a vector space over $\mathbb{F}$ with a bilinear map [,] : **L** × **L** → **L** such that following axioms are satisfied:

(i) $[a,a] = 0$ for all $a \in$ **L** and

(ii) $[a,[b,c]] + [b,[c,a]] + [c,[a,b]] = 0$ for all $a, b, c \in$ **L** (Jacobi Identity).

Remark 2.2. For a Lie algebra **L**, let $a, b \in$ **L**. By Definition 2.1 (i), we have

$$\begin{aligned} 0 &= [a+b,a+b] \\ &= [a,a+b] + [b,a+b] \\ &= [a,a] + [a,b] + [b,a] + [b,b] \\ &= [a,b] + [b,a]. \end{aligned}$$

Thus, $[a,b] = -[b,a]$.

Moreover, for any $a,b,c \in$ **L**, by Definition 2.1 (ii) we have

$$0 = [a,[b,c]] + [b,[c,a]] + [c,[a,b]].$$

Hence, $$[a,[b,c]] = -[c,[a,b]] - [b,[c,a]] = [[a,b],c] + [b,[a,c]].$$

Example 2.3. (14) Let **L** = $\mathbb{R}^3$ and $x = (x_1, x_2, x_3)$, $y = (y_1, y_2, y_3)$, $\in$ **L**. Define

$$[x,y] = (x_2y_3 - x_3y_2, x_3y_1 - x_1y_3, x_1y_2 - x_2y_1).$$

Then **L** is a Lie algebra over $\mathbb{F}$.

Definition 2.4. (9) A *(left) Leibniz algebra* **A** is a vector space over $\mathbb{F}$ with a bilinear map (called bracket) [,] : **A** × **A** → **A** that satisfies the Leibniz identity

$$[a,[b,c]] = [[a,b],c] + [b,[a,c]] \text{ for all } a, b, c \in \mathbf{A}.$$

It is easy to see that all Lie algebras are Leibniz algebras, as the Jacobi identity can be rearranged to match the Leibniz identity, as observed in Remark 2.2. For a Leibniz algebra **A**, if $[a,a] = 0$ for all $a \in$ **A**, then axiom (i) holds, and hence **A** is a Lie algebra. Note that Definition 2.4 is for left Leibniz algebras. One can define right Leibniz algebras in a similar way. Following Barnes (15), throughout this work, we will focus on left Leibniz algebras.

Example 2.5. Let **A** = span$\{x, y, z\}$ with non-zero brackets defined by $[x,x] = z$, $[x,y] = y$ and $[y,x] = -y$. Let $a, b, c \in$ **A** such that

$$a = \alpha_1 x + \alpha_2 y + \alpha_3 z,$$
$$b = \beta_1 x + \beta_2 y + \beta_3 z,$$
$$c = \gamma_1 x + \gamma_2 y + \gamma_3 z \qquad \text{for } \alpha_i, \beta_i, \gamma_i \in \mathbb{F}, 1 \le i \le 3.$$

Then

$$\begin{aligned}
[a,[b,c]] &= [a,[\beta_1 x + \beta_2 y + \beta_3 z, \gamma_1 x + \gamma_2 y + \gamma_3 z]] \\
&= [a,[\beta_1 x + \beta_2 y + \beta_3 z, \gamma_1 x] + [\beta_1 x + \beta_2 y + \beta_3 z, \gamma_2 y] \\
&\quad + [\beta_1 x + \beta_2 y + \beta_3 z, \gamma_3 z]] \\
&= [a,[\beta_1 x, \gamma_1 x] + [\beta_2 y, \gamma_1 x] + [\beta_3 z, \gamma_1 x] + [\beta_1 x, \gamma_2 y] + [\beta_2 y, \gamma_2 y] \\
&\quad + [\beta_3 z, \gamma_2 y] + [\beta_1 x, \gamma_3 z] + [\beta_2 y, \gamma_3 z] + [\beta_3 z, \gamma_3 z]] \\
&= [a, \beta_1\gamma_1[x,x] + \beta_2\gamma_1[y,x] + \beta_3\gamma_1[z,x] + \beta_1\gamma_2[x,y] + \beta_2\gamma_2[y,y] \\
&\quad + \beta_3\gamma_2[z,y] + \beta_1\gamma_3[x,z] + \beta_2\gamma_3[y,z] + \beta_3\gamma_3[z,z]] \\
&= [a,(\beta_1\gamma_1)z - (\beta_2\gamma_1)y + (\beta_1\gamma_2)y] \\
&= \beta_1\gamma_1[a,z] - \beta_2\gamma_1[a,y] + \beta_1\gamma_2[a,y] \\
&= \beta_1\gamma_1[\alpha_1 x + \alpha_2 y + \alpha_3 z, z] - \beta_2\gamma_1[\alpha_1 x + \alpha_2 y + \alpha_3 z, y] \\
&\quad + \beta_1\gamma_2[\alpha_1 x + \alpha_2 y + \alpha_3 z, y] \\
&= \beta_1\gamma_1(\alpha_1[x,z] + \alpha_2[y,z] + \alpha_3[z,z]) - \beta_2\gamma_1(\alpha_1[x,y] + \alpha_2[y,y] \\
&\quad + \alpha_3[z,y]) + \beta_1\gamma_2(\alpha_1[x,y] + \alpha_2[y,y] + \alpha_3[z,y]) \\
&= -(\alpha_1\beta_2\gamma_1)y + (\alpha_1\beta_1\gamma_2)y \\
&= (\alpha_1\beta_1\gamma_2 - \alpha_1\beta_2\gamma_1)y.
\end{aligned}$$

Similarly, we have

$$[[a,b],c] + [b,[a,c]] = [\alpha_1\beta_1 z - \alpha_2\beta_1 y + \alpha_1\beta_2 y, c] + [b,[\alpha_1\gamma_1 z - \alpha_2\gamma_1 y + \alpha_1\gamma_2 y]]$$

$= (\alpha_2\beta_1\gamma_1)y - (\alpha_1\beta_2\gamma_1)y - (\alpha_2\beta_1\gamma_1)y + (\alpha_1\beta_1\gamma_2)y$

$= (\alpha_1\beta_1\gamma_2 - \alpha_1\beta_2\gamma_1)y.$

Thus the Leibniz identity holds, hence, **A** is a Leibniz algebra. In fact, **A** is not a Lie algebra since $[x,x] = z \neq 0$.

For subsets M and N of a Leibniz algebra **A**, we define the *product* of M and N to be the subspace spanned by all brackets $[a,b]$, where $a \in M$ and $b \in N$, denoted by $[M,N]$.

Definition 2.6. (9) A subspace M of a Leibniz algebra **A** is called a *subalgebra* of **A** if $[M,M] \subseteq M$. A subspace M of a Leibniz algebra **A** is called an *ideal* of **A** if $[M, \mathbf{A}] \subseteq M$ and $[\mathbf{A}, M] \subseteq M$.

For ideals M and N of a Leibniz algebra **A**, there are several ways to construct new ideals from M and N, similar to the case of Lie algebras. The sum and intersection of two ideals of a Leibniz algebra are also ideals. However, the product of two ideals does not necessarily result in an ideal, as demonstrated below.

Example 2.7. (16) Let **A** = span$\{x, a, b, c, d\}$ with non-zero brackets defined by $[a,b] = c$, $[b,a] = d$, $[x,a] = a = -[a,x]$, $[x,c] = c$, $[x,d] = d$, $[c,x] = d$, $[d,x] = -d$. Let M = span$\{a, c, d\}$ and N = span$\{b, c, d\}$. Then M and N are ideals of **A**, but $[M,N]$ = span$\{c\}$ which is not an ideal of **A**.

Definition 2.8. (9) Let **A** be a Leibniz algebra. The *left center* of **A** is $Z^{\ell}(\mathbf{A}) = \{x \in \mathbf{A} \mid [x,a] = 0$ for all $a \in \mathbf{A}\}$. The *right center* of **A** is $Z^{r}(\mathbf{A}) = \{x \in \mathbf{A} \mid [a,x] = 0$ for all $a \in \mathbf{A}\}$. The *center* of **A** is $Z(\mathbf{A}) = Z^{\ell}(\mathbf{A}) \cap Z^{r}(\mathbf{A})$.

Given any Leibniz algebra **A**, we denote Leib(**A**) = span$\{[x,x] \mid x \in \mathbf{A}\}$. Clearly, Leib(**A**) = $\{0\}$ if and only if **A** is a Lie algebra.

Example 2.9. Consider the Leibniz algebra A = span$\{x, y, z\}$ with non-zero brackets defined by $[x,x] = z$, $[x,y] = y$ and $[y,x] = -y$. Then for all $a \in$ A,

$$[a,a] = [\alpha_1 x + \alpha_2 y + \alpha_3 z,\ \alpha_1 x + \alpha_2 y + \alpha_3 z] = 2\alpha_1 z + \alpha_1\, \alpha_2 y - \alpha_2 \gamma_1 y = 2\alpha_1 z.$$

We can see that A is not a Lie algebra because Leib(A) = span$\{z\}$.

Proposition 2.10. Let A be a Leibniz algebra. Then Z(A) and Leib(A) are ideals of A. Moreover, Leib(A) $\subseteq Z^{\ell}$(A).

Proof. Let $a \in Z$(A) and $b \in$ A. Then $[a,b] = 0 = [b,a]$. This implies that $[Z(\mathrm{A}),\mathrm{A}] \subseteq Z(\mathrm{A})$ and $[\mathrm{A},Z(\mathrm{A})] \subseteq Z(\mathrm{A})$, hence, Z(A) is an ideal of A. Let $x \in$ Leib(A) and $y \in$ A. Then there exist $u \in$ A and $\alpha \in \mathbb{F}$ such that $x = \alpha[u,u]$. Consider the element $[y + x, y + x] - [y,y] \in$ Leib(A), we have

$$\begin{aligned}
\mathrm{Leib(A)} \ni\ [y + x, y + x] - [y,y] &= [y, y + x] + [x, y + x] - [y,y] \\
&= [y,y] + [y,x] + [x, y + x] - [y,y] \\
&= [y,x] + [\alpha[u,u], y + x] \\
&= [y,x] + \alpha([u, [u, y + x]] - [u, [u, y + x]]) \\
&= [y,x].
\end{aligned}$$

Therefore, [A,Leib(A)] $\subseteq$ Leib(A). Moreover, $[x,y] = [\alpha[u,u],y] = \alpha([u,[u,y]] - [u,[u,y]]) = 0 \in$ Leib(A) for all $y \in$ A. Hence Leib(A) $\subseteq Z^{\ell}$(A) and Leib(A) is an ideal of A. □

For any ideal M of a Leibniz algebra A, we define the *quotient space* by A / M = $\{a + M \mid a \in \mathrm{A}\}$ with the bracket $[x + M, y + M] = [x,y] + M$, for all $x, y \in$ A.

Proposition 2.11. Let M be an ideal of Leibniz algebra A. Then A / M is a Leibniz algebra.

Proof. Observe that A / M is a subalgebra because for any $x, y \in$ A,

$$[x + M, y + M] = [x,y] + M \in \mathrm{A} / M.$$

Let $\alpha, \beta \in \mathbb{F}$ and $x, y, z \in$ A. Then

$$[\alpha x + \beta y + M, z + M] = [\alpha x + \beta y, z] + M = \alpha[x,z] + \beta[y,z] + M,$$

$$[x + M, \alpha y + \beta z + M] = [x, \alpha y + \beta z] + M = \alpha[x,y] + \beta[x,z] + M.$$

Thus, the bracket is bilinear. To check if the Leibniz identity is satisfied, we consider the following:

$$[x + M, [y + M, z + M]] = [x + M, [y,z] + M] = [x, [y,z]] + M.$$

We know that $[x,[y,z]] = [[x,y],z] + [y,[x,z]]$ since **A** is a Leibniz algebra. Thus the Leibniz identity holds. Now, we will check if the bracket is well defined. To do this, assume that $x + M = \tilde{x} + M$ and $y + M = \tilde{y} + M$. This implies that $\tilde{x} = x + i_1$ and $\tilde{y} = y + i_2$ for some $i_1, i_2 \in M$. Then

$$\begin{aligned} [\tilde{x} + M, \tilde{y} + M] &= [x + i_1 + M, y + i_2 + M] \\ &= [x + i_1, y + i_2] + M \\ &= [x,y] + [i_1,y] + [x,i_2] + [i_1,i_2] + M. \end{aligned}$$

Here, $[i_1,y]$, $[x,i_2]$ and $[i_1,i_2]$ are all in M since M is an ideal. Thus,

$$[x,y] + [i_1,y] + [x,i_2] + [i_1,i_2] + M = [x,y] + M = [x + M, y + M].$$

Therefore, the bracket is indeed well-defined and **A** / M is a Leibniz algebra. □

Proposition 2.12. Let **A** be a Leibniz algebra. Then Leib(**A**) is the minimal ideal of **A** such that **A** / Leib(**A**) is a Lie algebra.

Proof. Suppose there exists an ideal S such that **A** / S is a Lie algebra. Then S must have the property that for all $x \in$ **A**, $[x,x] \in S$. This is only achievable if $S = \{0\}$ or $S \supseteq$ Leib(**A**). Thus, Leib(**A**) is the minimal ideal of **A** such that **A** / Leib(**A**) is a Lie algebra. □

Definition 2.13. (9) Let **A** be a Leibniz algebra. A linear transformation δ: **A** → **A** is a *derivation* of **A** if $\delta[a,b] = [\delta(a),b] + [a, \delta(b)]$ for all $a, b \in$ **A**.

We denote Der(**A**) to be the set of all derivations of **A** with the commutator bracket $[\delta_1, \delta_2] := \delta_1 \circ \delta_2 - \delta_2 \circ \delta_1$ for any $\delta_1, \delta_2 \in$ Der(**A**).

Proposition 2.14. (14) Let **A** be a Leibniz algebra. Then Der(**A**) is a Lie algebra under the commutator bracket.

Proof. Let A be a Leibniz algebra. Since Der(A) is closed under linear combinations, it is a subspace of gl(A), the Lie algebra of all linear transformations on A under the commutator bracket. Let $\delta_1, \delta_2 \in$ Der(A). Then for all $x, y \in$ A we have

$$
\begin{aligned}
[\delta_1, \delta_2][x,y] &= \delta_1(\delta_2[x,y]) - \delta_2(\delta_1[x,y]) \\
&= \delta_1([\delta_2(x),y] + [x, \delta_2(y)]) - \delta_2([\delta_1(x),y] + [x, \delta_1(y)]) \\
&= \delta_1([\delta_2(x),y]) + \delta_1([x, \delta_2(y)]) - \delta_2([\delta_1(x),y]) - \delta_2([x, \delta_1(y)]) \\
&= [\delta_1(\delta_2(x)),y] + [\delta_2(x), \delta_1(y)] + [\delta_1(x), \delta_2(y)] + [x, \delta_1(\delta_2(y))] \\
&\quad - [\delta_2(\delta_1(x)),y] - [\delta_1(x), \delta_2(y)] - [\delta_2(x), \delta_1(y)] - [x, \delta_2(\delta_1(y))] \\
&= [\delta_1(\delta_2(x)),y] + [x, \delta_1(\delta_2(y))] - [\delta_2(\delta_1(x)),y] - [x, \delta_2(\delta_1(y))] \\
&= [\delta_1 \delta_2(x),y] + [x, \delta_1 \delta_2(y)] - [\delta_2 \delta_1(x),y] - [x, \delta_2 \delta_1(y)] \\
&= [[\delta_1, \delta_2](x),y] + [x,[\delta_1, \delta_2](y)]
\end{aligned}
$$

which implies that $[\delta_1, \delta_2] \in$ Der(A). Hence Der(A) is a subalgebra of the Lie algebra gl(A), and thus a Lie algebra. □

Definition 2.15. (9) Let A be a Leibniz algebra. An ideal M of A is a *characteristic ideal* if $\delta(M) \subseteq M$ for all $\delta \in$ Der(A).

As shown in (9), the ideals Leib(A) and Z^{ℓ}(A) are characteristic ideals. Let A be a Leibniz algebra. For any $a \in$ A, we define the *left multiplication* operator L_a : A $\rightarrow$ A by $L_a(b) = [a,b]$ for all $b \in$ A. Clearly, $L_a \in$ Der(A) because for all $b, c \in$ A we have $L_a[b,c] = [a,[b,c]] = [[a,b],c] + [b,[a,c]] = [L_a(b),c] + [b,L_a(c)]$.

For a Lie algebra L, a derivation d : L $\rightarrow$ L is inner if there exists $x \in$ L such that $d = \text{ad}_x$ where ad_x : L $\rightarrow$ L is defined by $\text{ad}_x(y) = [x,y]$ for all $y \in$ L. Several authors have adopted the same definition for inner derivations of Leibniz algebras. It is known that all derivations of simple Lie algebras are inner. However, as shown in (9) with this definition, there is a simple Leibniz algebra that contains an outer derivation. Hence we use the wider definition of inner derivations of Leibniz algebras given in (9).

Definition 2.16. (9) Let **A** be a Leibniz algebra. A derivation $\delta \in \mathrm{Der}(\mathbf{A})$ is said to be *inner* if there exists $x \in \mathbf{A}$ such that $\mathrm{im}(\delta - L_x) \subseteq \mathrm{Leib}(\mathbf{A})$.

Example 2.17. Consider the Leibniz algebra **A** with the ordered basis $B = \{x, y, z\}$ and non-zero brackets defined by $[x,x] = z$, $[x,y] = y$ and $[y,x] = -y$. Let $\delta \in \mathrm{Der}(\mathbf{A})$ and define the action of δ on the basis elements as follows:

$$\delta(x) = \alpha_1 x + \alpha_2 y + \alpha_3 z,$$

$$\delta(y) = \beta_1 x + \beta_2 y + \beta_3 z \quad \text{and}$$

$$\delta(z) = \gamma_1 x + \gamma_2 y + \gamma_3 z \quad \text{for } \alpha_i, \beta_i, \gamma_i \in \mathbb{F}, 1 \le i \le 3.$$

Therefore, $[\delta]_B = \begin{bmatrix} \alpha_1 & \beta_1 & \gamma_1 \\ \alpha_2 & \beta_2 & \gamma_2 \\ \alpha_3 & \beta_3 & \gamma_3 \end{bmatrix}$. By the derivation property, we have

$$\begin{aligned} \delta[x,x] &= [\delta(x),x] + [x, \delta(x)] \\ &= [\alpha_1 x + \alpha_2 y + \alpha_3 z, x] + [x, \alpha_1 x + \alpha_2 y + \alpha_3 z] \\ &= \alpha_1[x,x] + \alpha_2[y,x] + \alpha_1[x,x] + \alpha_2[x,y] \\ &= \alpha_1 z - \alpha_2 y + \alpha_1 z + \alpha_2 y \\ &= 2\alpha_1 z. \end{aligned}$$

Since $[x,x] = z$, $2\alpha_1 z = \delta[x,x] = \delta(z) = \gamma_1 x + \gamma_2 y + \gamma_3 z$. Then we have $\gamma_1 = \gamma_2 = 0$ and $\gamma_3 = 2\alpha_1$. Similarly, we have

$$\begin{aligned} \beta_1 x + \beta_2 y + \beta_3 z = \delta(y) = \delta[x,y] &= [\delta(x),y] + [x, \delta(y)] \\ &= [\alpha_1 x + \alpha_2 y + \alpha_3 z, y] + [x, \beta_1 x + \beta_2 y + \beta_3 z] \\ &= \alpha_1 [x,y] + \beta_1 [x,x] + \beta_2 [x,y] \\ &= \alpha_1 y + \beta_1 z + \beta_2 y. \end{aligned}$$

Then we have $\beta_1 = \beta_3$ and $\alpha_1 = 0$. It follows that $\gamma_3 = 0$. Also, we have

$$\begin{aligned} 0 = \delta(0) = \delta[x,z] &= [\delta(x),z] + [x, \delta(z)] \\ &= [\alpha_1 x + \alpha_2 y + \alpha_3 z, z] + [x, \gamma_1 x + \gamma_2 y + \gamma_3 z] \\ &= 0 + \gamma_1 [x,x] + \gamma_2 [x,y] \\ &= \gamma_1 z + \gamma_2 y \end{aligned}$$

which implies $\gamma_1 = \gamma_2 = 0$. In addition, we have

$$
\begin{aligned}
-\beta_1 x - \beta_2 y - \beta_3 z &= -(\beta_1 x + \beta_2 y + \beta_3 z)\\
&= \delta(-y)\\
&= \delta[y,x]\\
&= [\delta(y),x] + [y, \delta(x)]\\
&= [\beta_1 x + \beta_2 y + \beta_3 z, x] + [y, \alpha_1 x + \alpha_2 y + \alpha_3 z]\\
&= \beta_1[x,x] + \beta_2[y,x] + \alpha_1[y,x]\\
&= \beta_1 z - \beta_2 y - \beta_3 y
\end{aligned}
$$

which implies $\beta_1 = \beta_3 = 0$.

$$
\begin{aligned}
0 = \delta(0) &= \delta[z,x]\\
&= [\delta(z),x] + [z, \delta(x)]\\
&= [\gamma_1 x + \gamma_2 y + \gamma_3 z, x] + [z, \alpha_1 x + \alpha_2 y + \alpha_3 z]\\
&= \gamma_1 z - \gamma_2 y
\end{aligned}
$$

which implies $\gamma_1 = \gamma_2 = 0$.

Therefore, $[\delta]_B = \begin{bmatrix} 0 & 0 & 0 \\ \alpha_2 & \beta_2 & 0 \\ \alpha_3 & 0 & 0 \end{bmatrix}$

$$= \alpha_2 \begin{bmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} + \alpha_3 \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{bmatrix} + \beta_2 \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

Hence Der(**A**) = span{ δ_1, δ_2, δ_3}, where

$\delta_1(x) = y$, $\delta_1(y) = 0$, $\delta_1(z) = 0$,

$\delta_2(x) = z$, $\delta_2(y) = 0$, $\delta_2(z) = 0$,

$\delta_3(x) = 0$, $\delta_3(y) = y$, $\delta_3(z) = 0$.

Since im($\delta_1 - L_{-y}$) ⊆ Leib(**A**), im($\delta_2 - L_z$) ⊆ Leib(**A**) and im($\delta_3 - L_x$) ⊆ Leib(**A**), δ_1, δ_2 and δ_3 are inner.

For a Leibniz algebra **A**, we define the ideals $\mathbf{A}^{(1)} = \mathbf{A} = \mathbf{A}^1$, $\mathbf{A}^{(i)} = [\mathbf{A}^{(i-1)}, \mathbf{A}^{(i-1)}]$ and $\mathbf{A}^i = [\mathbf{A}, \mathbf{A}^{i-1}]$ for $i \in \mathbb{Z}_{\geq 2}$. The Leibniz algebra is said to be *solvable* (resp. *nilpotent*) if $\mathbf{A}^{(m)} = \{0\}$ (resp. $\mathbf{A}^m = \{0\}$) for some positive integer m. The *maximal solvable* (resp. *nilpotent*) ideal of **A** is called the *radical* (resp. *nilradical*) denoted by rad(**A**) (resp. nilrad(**A**)). A Leibniz algebra **A** is called *simple* if its ideals are only {0}, Leib(**A**), **A** and [**A**, **A**] ≠

Leib(A). A Leibniz algebra A is *semisimple* if rad(A) = Leib(A). We recall an analog of Levi's theorem for Leibniz algebras which will be used in this work.

Theorem 2.18. (17) Let A be a Leibniz algebra. Then there exists a subalgebra S (which is a semisimple Lie algebra) of A such that A = S + rad(A) and $S \cap$ rad(A) = {0}.

Corollary 2.19. Let A be a semisimple Leibniz algebra. Then there exists a semisimple Lie algebra S of A such that A = S + Leib(A).

CHAPTER 3
DECOMPOSITIONS OF LEIBNIZ ALGEBRS

Let IDer(A) be the set of all inner derivations of a Leibniz algebra A and L(A) = span{L_a | a ∈ A}. It should be noted that L(A) ⊆ IDer(A) ⊆ Der(A).

Proposition 3.1. Let A be a Leibniz algebra. Then L(A) and IDer(A) are ideals of Der(A).

Proof. Let $L_a \in L(A)$ where $a \in A$ and $d \in Der(A)$, we have $[L_a,d](x) = L_a(d(x)) - d(L_a(x)) = [a,d(x)] - d[a,x] = [a,d(x)] - [d(a),x] - [a,d(x)] = L_{-d(a)}(x)$ for all $x \in A$. Then $[L_a,d] = L_{-d(a)}$. Hence, L(A) is an ideal of Der(A). To show that IDer(A) is an ideal of Der(A). Let $d \in$ Der(A) and $\delta \in$ IDer(A). Then there exist $b \in A$ such that im($\delta - L_b$) ⊆ Leib(A). For any $x \in$ A, we have

$$\begin{aligned}[\delta,d](x) &= \delta(d(x)) - d(\delta(x)) \\ &= \delta(d(x)) - L_b(d(x)) + [b,d(x)] - d(\delta(x)) \\ &= (\delta - L_b)(d(x)) + d([b,x]) - [d(b),x] - d(\delta(x)).\end{aligned}$$

Consider,
$$\begin{aligned}[\delta,d](x) + [d(b),x] &= (\delta - L_b)(d(x)) + d(L_b(x)) - d(\delta(x)) \\ &= (\delta - L_b)(d(x)) - d((\delta - L_b)(x)) \\ &\in \text{Leib}(A).\end{aligned}$$

Thus, im($[\delta,d] - L_{-d(b)}$) ⊆ Leib(A) which implies that $[\delta,d] \in$ IDer(A). Similary, $[d,\delta] \in$ IDer(A). Hence IDer(A) is an ideal of Der(A). □

By studying the set of left multiplications and the set of all inner derivations of the Leibniz algebra A, we became interested in the elements x such that the left multiplication L_x maps from A into Leib(A). So, we define the set $I_A = \{x \in A \mid \text{im}(L_x) \subseteq \text{Leib}(A)\}$. It is clear that Leib(A) ⊆ I_A. The following are easy but important observations.

Proposition 3.2. (18) Let A be a Leibniz algebra. Then I_A is a characteristic ideal of A.

Proof. To show that I_A is an ideal of A, let $x \in I_A$ and $a \in A$. Then for all $y \in A$, $L_{[x,a]}(y) = [[x,a],y] \in$ Leib(A) and $L_{[a,x]}(y) = [[a,x],y] = [a,[x,y]] - [x,[a,y]] \in$ Leib(A), hence $[x,a]$, $[a,x] \in I_A$ which implies that I_A is an ideal of A. To show that I_A is a characteristic ideal, let $x \in I_A$

and $d \in \mathrm{Der}(\mathsf{A})$. Then for all $y \in \mathsf{A}$, $L_{d(x)}(y) = [d(x),y] = d[x,y] - [x,d(y)] = d(L_x(y)) - L_x(d(y)) \in \mathrm{Leib}(\mathsf{A})$ and hence $d(x) \in I_{\mathsf{A}}$. This proves that I_{A} is a characteristic ideal of A. □

Proposition 3.3. (18) Let A be a Leibniz algebra. Then $Z^{\ell}(\mathsf{A} / \mathrm{Leib}(\mathsf{A})) \cong I_{\mathsf{A}} / \mathrm{Leib}(\mathsf{A})$.

Proof. Clearly, Leib(A) is an ideal of I_{A}. Then $Z^{\ell}(\mathsf{A} / \mathrm{Leib}(\mathsf{A})) = \{x + \mathrm{Leib}(\mathsf{A}) \mid [x + \mathrm{Leib}(\mathsf{A}), y + \mathrm{Leib}(\mathsf{A})] = \mathrm{Leib}(\mathsf{A}) \text{ for all } y \in \mathsf{A}\} = \{x + \mathrm{Leib}(\mathsf{A}) \mid [x,y] \in \mathrm{Leib}(\mathsf{A}) \text{ for all } y \in \mathsf{A}\}$. By the trivial isomorphism φ defined by $\varphi(x + \mathrm{Leib}(\mathsf{A})) = x + \mathrm{Leib}(\mathsf{A})$ for all $x + \mathrm{Leib}(\mathsf{A}) \in Z^{\ell}(\mathsf{A} / \mathrm{Leib}(\mathsf{A}))$, it follows that $Z^{\ell}(\mathsf{A} / \mathrm{Leib}(\mathsf{A})) \cong I_{\mathsf{A}} / \mathrm{Leib}(\mathsf{A})$. □

It is known that $L(\mathsf{A})$ forms a Lie algebra under the commutator bracket. The following result is easily derived.

Theorem 3.4. (18) Let A be a Leibniz algebra. Then $\mathsf{A} / Z^{\ell}(\mathsf{A}) \cong L(\mathsf{A})$.

Proof. Define $\varphi : \mathsf{A} \to L(\mathsf{A})$ by $\varphi(x) = L_x$ for all $x \in \mathsf{A}$. Then for any $x, y, z \in \mathsf{A}$, we have $\varphi([x,y])(z) = L_{[x,y]}(z) = [[x,y],z]$ and $[\varphi(x), \varphi(y)](z) = [L_x,L_y](z) = L_xL_y(z) - L_yL_x(z) = [x,[y,z]] - [y,[x,z]] = [[x,y],z] + [y,[x,z]] - [y,[x,z]] = [[x,y],z]$. Therefore, $\varphi([x,y]) = [\varphi(x), \varphi(y)]$. Clearly, φ is onto and $\ker(\varphi) = \{x \in \mathsf{A} \mid L_x = 0\} = \{x \in \mathsf{A} \mid [x,y] = 0 \text{ for all } y \in \mathsf{A}\} = Z^{\ell}(\mathsf{A})$. Hence, $\mathsf{A} / Z^{\ell}(\mathsf{A}) \cong L(\mathsf{A})$. □

The following is immediate from Proposition 3.3 and Theorem 3.4.

Corollary 3.5. Let A be a Leibniz algebra. Then $\mathsf{A} / I_{\mathsf{A}} \cong L(\mathsf{A} / \mathrm{Leib}(\mathsf{A}))$.

In addition to the elements that render the image of its left multiplication a subset of Leib(A), we also investigate the set of all derivations of a Leibniz algebra A whose image resides within Leib(A). We define the set $I = \{d \in \mathrm{Der}(\mathsf{A}) \mid \mathrm{im}(d) \subseteq \mathrm{Leib}(\mathsf{A})\}$. Clearly, $I \subseteq \mathrm{IDer}(\mathsf{A}) \subseteq \mathrm{Der}(\mathsf{A})$.

Lemma 3.6. Let A be a Leibniz algebra. Then I is an ideal of Der(A).

Proof. Let $d \in I$ and $\delta \in$ Der(A). Then im(d) $\subseteq$ Leib(A). Since Leib(A) is a characteristic ideals of A, for any $x \in$ A, $[d, \delta](x) = d(\delta(x)) - \delta(d(x)) \in$ Leib(A). This implies that [I,Der(A)] $\subseteq I$. Hence I is a ideal of Der(A). $\square$

The following theorem is one of our main results.

Theorem 3.7. (18) Let A be a Leibniz algebra. Then IDer(A) is an ideal of Der(A) and IDer(A) = L(A) + I. Moreover, if Z(A / Leib(A)) is trivial, then L(A) ∩ I = {0}.

Proof. Let $d \in$ IDer(A). Then there exists $x \in$ A such that im($d - L_x$) $\subseteq$ Leib(A). Then $d - L_x \in I$ and hence $d \in L$(A) + I. This implies that IDer(A) $\subseteq L$(A) + I. Since the reverse inclusion is clear, we have IDer(A) = L(A)+I. Consequently, IDer(A) is an ideal of Der(A). Note that L(Leib(A)) = $\{L_a \mid a \in$ Leib(A)$\}$ = {0} because Leib(A) $\subseteq Z^{\ell}$(A). Suppose that Z(A / Leib(A)) is trivial. Let $L_x \in L$(A) ∩ I. Then $[x,a] \in$ Leib(A) for all $a \in$ A. Thus x + Leib(A) $\in Z$(A / Leib(A)) which implies that $x \in$ Leib(A). Therefore, L(A) ∩ $I \subseteq L$(Leib(A)) = {0}. $\square$

Example 3.8. Consider the Leibniz algebra A = span$\{w, x, y, z\}$ with non-zero multiplications defined by $[w,w] = y$ and $[x,w] = z$. Clearly, Leib(A) = span$\{y, z\}$. By direct calculation, we have that Der(A) = span$\{d_1, d_2, d_3, d_4, d_5, d_6, d_7\}$ where

$d_1(w) = w,$	$d_1(x) = 0,$	$d_1(y) = 2y,$	$d_1(z) = z,$
$d_2(w) = x,$	$d_2(x) = 0,$	$d_2(y) = z,$	$d_2(z) = 0,$
$d_3(w) = y,$	$d_3(x) = 0,$	$d_3(y) = 0,$	$d_3(z) = 0,$
$d_4(w) = z,$	$d_4(x) = 0,$	$d_4(y) = 0,$	$d_4(z) = 0,$
$d_5(w) = 0,$	$d_5(x) = x,$	$d_5(y) = 0,$	$d_5(z) = z,$
$d_6(w) = 0,$	$d_6(x) = y,$	$d_6(y) = 0,$	$d_6(z) = 0,$
$d_7(w) = 0,$	$d_7(x) = z,$	$d_7(y) = 0,$	$d_7(z) = 0.$

Then we have L(A) = span{d_3, d_4} and I = span{d_3, d_4, d_6, d_7}. Hence IDer(A) = span{d_3, d_4, d_6, d_7} = L(A) + I. Note that Z(A / Leib(A)) = span{w + Leib(A), x + Leib(A)} and L(A) ∩ I = span{d_3, d_4} in this case.

Example 3.9. Consider the Leibniz algebra A = span{x, y, z} with non-zero multiplications defined by $[x,y] = y$, $[y,x] = -y$ and $[x,x] = z$. In this case, we have Leib(A) = span{z} = Z(A) and Z(A / Leib(A)) is trivial. By direct calculation, we have that Der(A) = span{d_1, d_2, d_3} = IDer(A) where

$$d_1(x) = y, \quad d_1(y) = 0, \quad d_1(z) = 0,$$

$$d_2(x) = z, \quad d_2(y) = 0, \quad d_2(z) = 0,$$

$$d_3(x) = 0, \quad d_3(y) = y, \quad d_3(z) = 0.$$

Then we have L(A) = span{d_1, $d_2 + d_3$} and I = span{d_2}. Hence IDer(A) = L(A) + I and L(A) ∩ I = {0}.

Example 3.10. Consider the Leibniz algebra A = span{x, y, z} with non-zero multiplications defined by $[x,y] = y$,$[y,x] = -y$ and $[x,z] = z$. Clearly, Leib(A) = span{z}, Z(A) = {0} and Z(A / Leib(A)) is trivial. By direct calculation, we have that Der(A) = span{d_1, d_2, d_3} = IDer(A) where

$$d_1(x) = y, \quad d_1(y) = 0, \quad d_1(z) = 0,$$

$$d_2(x) = 0, \quad d_2(y) = 0, \quad d_2(z) = z,$$

$$d_3(x) = 0, \quad d_3(y) = y, \quad d_3(z) = 0.$$

Then we have L(A) = span{d_1, $d_2 + d_3$} and I = span{d_2}. Hence, IDer(A) = L(A) + I and L(A) ∩ I = {0} in this case.

Following the definition of the holomorph of a Lie algebra, the holomorph of the Leibniz algebra A is defined to be the vector space hol(A) := A ⊕ Der(A), with the bracket defined by $[x + \delta_1, y + \delta_2] = [x,y] + \delta_1(y) + [L_x, \delta_2] + [\delta_1, \delta_2]$ for all $x, y \in$ A and $\delta_1, \delta_2 \in$ Der(A) (see (9)). By direct calculation, it is known that hol(A) is a Leibniz algebra.

Proposition 3.11. Let A be a Leibniz algebra. Then

$$\mathrm{hol}(A) / (I_A \oplus I) \cong A / I_A \oplus \mathrm{Der}(A) / I.$$

Proof. Since I_A and I are ideals of A and Der(A), respectively, we have $I_A \oplus I$ is an ideal of hol(A). By the trivial isomorphism φ defined by $\varphi(x + \delta + I_A \oplus I) = x + I_A + \delta + I$ for all $x + \delta \in \mathrm{hol}(A)$, it follows that $\mathrm{hol}(A) / (I_A \oplus I) \cong A / I_A \oplus \mathrm{Der}(A) / I$. □

For two subspaces M and N of hol(A), the *left centralizer* of M in N is defined to be $Z^{\ell}_{N}(M) = \{x \in N \mid [x,M] = 0\}$. The following results were obtained in (9).

Proposition 3.12. (9) Let A be a Leibniz algebra. Then $Z^{\ell}_{\mathrm{hol}(A)}(A) = \{x - L_x \mid x \in A\}$.

Proposition 3.13. (9) Let A be a Leibniz algebra. Then $A \cap Z^{\ell}_{\mathrm{hol}(A)}(A) = Z^{\ell}(A)$.

Then the following follows immediately from Proposition 3.3 and Proposition 3.13.

Corollary 3.14. Let A be a Leibniz algebra. Then

$$A / \mathrm{Leib}(A) \cap Z^{\ell}_{\mathrm{hol}(A/\mathrm{Leib}(A))}(A/\mathrm{Leib}(A)) \cong I_A / \mathrm{Leib}(A).$$

Next, we study properties of the decompositions of Leibniz algebras. These results will also be useful for proving properties of complete Leibniz algebras in the next chapter. We assume the Leibniz algebra A is the direct sum of two ideals, i.e., $A = A_1 \oplus A_2$ where A_1 and A_2 are ideals of A. In (6), Meng proved that for a Lie algebra L if $L = L_1 \oplus L_2$ where L_1 and L_2 are ideals of L, then $Z(L) = Z(L_1) \oplus Z(L_2)$. Moreover, $\mathrm{Der}(L) = \mathrm{Der}(L_1) \oplus \mathrm{Der}(L_2)$ if $Z(L) = \{0\}$. In the following theorems we obtain some analogous results for Leibniz algebras.

Theorem 3.15. Let the Leibniz algebra $\mathrm{A} = \mathrm{A}_1 \oplus \mathrm{A}_2$ where A_1 and A_2 are ideals of A. Then

(i) $\mathrm{Leib}(\mathrm{A}) = \mathrm{Leib}(\mathrm{A}_1) \oplus \mathrm{Leib}(\mathrm{A}_2)$,

(ii) For any $a_1 \in \mathrm{A}_1$ and $a_2 \in \mathrm{A}$, if $a_1 + a_2 \in \mathrm{Leib}(\mathrm{A})$, then $a_1 \in \mathrm{Leib}(\mathrm{A}_1)$ and $a_2 \in \mathrm{Leib}(\mathrm{A}_2)$,

(iii) $Z(\mathrm{A}) = Z(\mathrm{A}_1) \oplus Z(\mathrm{A}_2)$,

(iv) $L(\mathrm{A}) = L(\mathrm{A}_1) \oplus L(\mathrm{A}_2)$,

(v) $\mathrm{A}^2 = \mathrm{A}_1^2 \oplus \mathrm{A}_2^2$,

(vi) $I_{\mathrm{A}} = I_{\mathrm{A}_1} \oplus I_{\mathrm{A}_2}$.

Proof. (i) If $a \in \mathrm{Leib}(\mathrm{A}_1) \cap \mathrm{Leib}(\mathrm{A}_2)$, then $a \in \mathrm{A}_1 \cap \mathrm{A}_2 = \{0\}$ hence $a = 0$ which implies $\mathrm{Leib}(\mathrm{A}_1) \cap \mathrm{Leib}(\mathrm{A}_2) = \{0\}$. Let $a \in \mathrm{A}$. Then there exist $a_1 \in \mathrm{A}_1$ and $a_2 \in \mathrm{A}_2$ such that $a = a_1 + a_2$. Since $[\mathrm{A}_1,\mathrm{A}_2], [\mathrm{A}_2,\mathrm{A}_1] \subseteq \mathrm{A}_1 \cap \mathrm{A}_2 = \{0\}$, we have that

$$\begin{aligned}
[a,a] &= [a_1 + a_2, a_1 + a_2] \\
&= [a_1,a_1] + [a_1,a_2] + [a_2,a_1] + [a_2,a_2] \\
&= [a_1,a_1] + [a_2,a_2] \\
&\in \mathrm{Leib}(\mathrm{A}_1) + \mathrm{Leib}(\mathrm{A}_2).
\end{aligned}$$

Hence $\mathrm{Leib}(\mathrm{A}) = \mathrm{span}\{[a,a] \mid a \in \mathrm{A}\} \subseteq \mathrm{Leib}(\mathrm{A}_1) + \mathrm{Leib}(\mathrm{A}_2)$. Since the reverse inclusion is clear, we have that $\mathrm{Leib}(\mathrm{A}) = \mathrm{Leib}(\mathrm{A}_1) \oplus \mathrm{Leib}(\mathrm{A}_2)$.

(ii) Let $a_1 \in \mathrm{A}_1$ and $a_2 \in \mathrm{A}_2$. Assume that $a_1 + a_2 \in \mathrm{Leib}(\mathrm{A})$. If $a_1 + a_2 = 0$, then $a_1 = -a_2 \in \mathrm{A}_2$. Since $\mathrm{A}_1 \cap \mathrm{A}_2 = \{0\}$, it follows that $a_1 = a_2 = 0$. Suppose that $a_1 + a_2 \neq 0$. By (i), $a_1 + a_2 \in \mathrm{Leib}(\mathrm{A}) = \mathrm{Leib}(\mathrm{A}_1) \oplus \mathrm{Leib}(\mathrm{A}_2)$. There exist $b_1 \in \mathrm{Leib}(\mathrm{A}_1)$ and $b_2 \in \mathrm{Leib}(\mathrm{A}_2)$ such that $a_1 + a_2 = b_1 + b_2$. Thus, $a_1 - b_1 = b_2 - a_2 \in \mathrm{A}_1 \cap \mathrm{A}_2 = \{0\}$. Hence $a_1 = b_1 \in \mathrm{Leib}(\mathrm{A}_1)$ and $a_2 = b_2 \in \mathrm{Leib}(\mathrm{A}_2)$.

(iii) Since $\mathrm{A}_1 \cap \mathrm{A}_2 = \{0\}$, $Z(\mathrm{A}_1) \cap Z(\mathrm{A}_2) = \{0\}$. Let $a_1 \in Z(\mathrm{A}_1)$ and $a_2 \in Z(\mathrm{A}_2)$. Let $b \in \mathrm{A}$. Then there exist $b_1 \in \mathrm{A}_1$ and $b_2 \in \mathrm{A}_2$ such that $b = b_1 + b_2$. Then we have

$$\begin{aligned}
[a_1 + a_2, b] &= [a_1 + a_2, b_1 + b_2] \\
&= [a_1,b_1] + [a_2,b_1] + [a_1,b_2] + [a_2,b_2] \\
&= [a_1,b_1] + [a_2,b_2] && (\because [\mathrm{A}_1,\mathrm{A}_2] = \{0\}) \\
&= 0. && (\because a_1 \in Z(\mathrm{A}_1) \text{ and } a_2 \in Z(\mathrm{A}_2)).
\end{aligned}$$

Similarly, $[b, a_1 + a_2] = [b_1 + b_2, a_1 + a_2] = [b_1,a_1] + [b_2,a_2] = 0$. Then $a_1 + a_2 \in Z(\mathrm{A})$. Hence $Z(\mathrm{A}_1) \oplus Z(\mathrm{A}_2) \subseteq Z(\mathrm{A})$. To show that $Z(\mathrm{A}) \subseteq Z(\mathrm{A}_1) \oplus Z(\mathrm{A}_2)$, let $a \in Z(\mathrm{A})$. Then there exist $a_1 \in \mathrm{A}_1$ and $a_2 \in \mathrm{A}_2$ such that $a = a_1 + a_2$. For any $b_1 \in \mathrm{A}_1$, we have $[a_1,b_1] = [a - a_2, b_1] = [a,b_1] - [a_2,b_1] = 0$ and $[b_1,a_1] = [b_1, a - a_2] = [b_1,a] - [b_1,a_2] = 0$ because $a \in Z(\mathrm{A})$ and $[\mathrm{A}_1,\mathrm{A}_2] = \{0\}$. Hence $a_1 \in Z(\mathrm{A}_1)$. Similarly, we have $a_2 \in Z(\mathrm{A}_2)$. Therefore, $Z(\mathrm{A}) = Z(\mathrm{A}_1) \oplus Z(\mathrm{A}_2)$.

(iv) Let $a \in \mathrm{A}$. Then $L_a \in L(\mathrm{A})$ and there exist $a_1 \in \mathrm{A}_1$ and $a_2 \in \mathrm{A}_2$ such that $a = a_1 + a_2$. Thus, for any $x \in \mathrm{A}$, we have $L_a(x) = L_{a_1+a_2}(x) = [a_1 + a_2, x] = [a_1,x] + [a_2,x] = L_{a_1}(x) + L_{a_2}(x) = L_{a_1}(x_1 + x_2) + L_{a_2}(x_1 + x_2) = L_{a_1}(x_1) + L_{a_2}(x_2)$ for some $x_1 \in \mathrm{A}_1$ and $x_2 \in \mathrm{A}_2$. This implies that $L_a \in L(\mathrm{A}_1) + L(\mathrm{A}_2)$. It is clear that $L(\mathrm{A}_1) + L(\mathrm{A}_2) \subseteq L(\mathrm{A})$ and $L(\mathrm{A}_1) \cap L(\mathrm{A}_2) = \{0\}$. Hence $L(\mathrm{A}) = L(\mathrm{A}_1) \oplus L(\mathrm{A}_2)$.

(v) Let $a, b \in \mathrm{A}$ and $\alpha \in \mathbb{F}$. Since $\mathrm{A} = \mathrm{A}_1 \oplus \mathrm{A}_2$, there exist $a_1, b_1 \in \mathrm{A}_1$ and $a_2, b_2 \in \mathrm{A}_2$ such that $a = a_1 + a_2$ and $b = b_1 + b_2$. Then $\alpha[a,b] = \alpha[a_1 + a_2, b_1 + b_2] = [\alpha a_1 + \alpha a_2, b_1 + b_2] = [\alpha a_1,b_1] + [\alpha a_1,b_2] + [\alpha a_2,b_1] + [\alpha a_2,b_2] = [\alpha a_1,b_1] + [\alpha a_2,b_2] \in [\mathrm{A}_1, \mathrm{A}_1] + [\mathrm{A}_2, \mathrm{A}_2] = \mathrm{A}_1^2 + \mathrm{A}_2^2$. Thus, $\mathrm{A}^2 \subseteq \mathrm{A}_1^2 + \mathrm{A}_2^2$. Clearly, $\mathrm{A}_1^2 + \mathrm{A}_2^2 \subseteq \mathrm{A}^2$ and $\mathrm{A}_1^2 \cap \mathrm{A}_2^2 = \{0\}$. Hence $\mathrm{A}^2 = \mathrm{A}_1^2 \oplus \mathrm{A}_2^2$.

(vi) Observe that $I_{\mathrm{A}_1} + I_{\mathrm{A}_2} \subseteq I_{\mathrm{A}}$ and $I_{\mathrm{A}_1} \cap I_{\mathrm{A}_2} = \{0\}$. Let $x \in I_{\mathrm{A}}$. Since $x \in \mathrm{A}$, there exist $x_1 \in \mathrm{A}_1$ and $x_2 \in \mathrm{A}_2$ such that $x = x_1 + x_2$. It follows that $L_{x_1}(\mathrm{A}) + L_{x_2}(\mathrm{A}) = L_{x_1+x_2}(\mathrm{A}) = L_x(\mathrm{A}) \subseteq \mathrm{Leib}(\mathrm{A}) = \mathrm{Leib}(\mathrm{A}_1) \oplus \mathrm{Leib}(\mathrm{A}_2)$. By (ii), we have $L_{x_i}(\mathrm{A}) \subseteq \mathrm{Leib}(\mathrm{A}_i)$ for $i = 1, 2$. Hence $I_{\mathrm{A}} = I_{\mathrm{A}_1} \oplus I_{\mathrm{A}_2}$.

□

Corollary 3.16. Let the Leibniz algebra $\mathrm{A} = \mathrm{A}_1 \oplus \mathrm{A}_2$ where A_1 and A_2 are ideals of A. Then

$$Z(\mathrm{A} / \mathrm{Leib}(\mathrm{A})) = \{0\} \text{ if and only if } Z(\mathrm{A}_i / \mathrm{Leib}(\mathrm{A}_i)) = \{0\} \text{ for all } i = 1, 2.$$

Proof. By Theorem 3.15, we have

$$Z(\mathrm{A} / \mathrm{Leib}(\mathrm{A})) = Z\left(\frac{\mathrm{A}_1 \oplus \mathrm{A}_2}{\mathrm{Leib}(\mathrm{A}_1) \oplus \mathrm{Leib}(\mathrm{A}_2)}\right) \cong Z(\mathrm{A}_1 / \mathrm{Leib}(\mathrm{A}_1)) \oplus Z(\mathrm{A}_2 / \mathrm{Leib}(\mathrm{A}_2))$$

Hence the result follows. □

Let the Leibniz algebra $\mathrm{A} = \mathrm{A}_1 \oplus \mathrm{A}_2$ where A_1 and A_2 are ideals of A. For $\delta \in \mathrm{Der}(\mathrm{A}_1)$, we can extend δ to be a derivation on A by defining $\delta(x_1 + x_2) = \delta(x_1)$ for any $x_1 \in \mathrm{A}_1$ and $x_2 \in \mathrm{A}_2$. Similarly, for $\delta \in \mathrm{Der}(\mathrm{A}_2)$, we can extend δ to be a derivation on A by defining $\delta(x_1 + x_2) = \delta(x_2)$ for any $x_1 \in \mathrm{A}_1$ and $x_2 \in \mathrm{A}_2$. Hence, we can and do consider $\delta \in$

Der(A_1) and $\delta \in$ Der(A_2) as derivations on A and view Der(A_i) $\subseteq$ Der(A) for i = 1, 2. The following theorem is one of our main results.

Theorem 3.17. Let the Leibniz algebra $A = A_1 \oplus A_2$ where A_1 and A_2 are ideals of A. Then

$$\text{Der}(A) = (\text{Der}(A_1) + I_1) \oplus (\text{Der}(A_2) + I_2$$

where $I_1 = \{ \delta \in \text{Der}(A) \mid \delta(A_2) = \{0\} \text{ and } \delta(A_1) \subseteq A_2 \cap Z(A)\}$ and

$I_2 = \{ \delta \in \text{Der}(A) \mid \delta(A_1) = \{0\} \text{ and } \delta(A_2) \subseteq A_1 \cap Z(A)\}$.

Proof. First we observe that if $\delta \in (\text{Der}(A_1) + I_1) \cap (\text{Der}(A_2) + I_2)$, then $\delta \in (\text{Der}(A_i) + I_i), i$ = 1, 2. So $\delta(A) \subseteq A_1 \cap A_2 = \{0\}$ which implies $\delta = 0$ and hence $(\text{Der}(A_1) \oplus I_1) \cap (\text{Der}(A_2) \oplus I_2)$ = {0}. To show that $\text{Der}(A) \subseteq (\text{Der}(A_1) + I_1) \oplus (\text{Der}(A_2) + I_2)$, let $0 \neq \delta \in \text{Der}(A)$. Suppose there exists $x \in A_1$ such that $0 \neq \delta(x) \in A_2$. Then we have that $[\delta(x),x_1] = 0 = [x_1, \delta(x)]$ for all $x_1 \in A_1$. Thus, $\delta(x) \in Z(A_1) \subseteq Z(A)$ which implies that $\delta(x) \in Z(A) \cap A_2$. Similarly, if there exists $x \in A_2$ such that $0 \neq \delta(x) \in A_1$, then $\delta(x) \in Z(A) \cap A_1$. Set

$$S_{11} = \{x_1 \in A_1 \mid \delta(x_1) \in A_1\},$$

$$S_{12} = \{x_1 \in A_1 \mid \delta(x_1) \in A_2\},$$

$$S_{21} = \{x_2 \in A_2 \mid \delta(x_2) \in A_1\},$$

$$S_{22} = \{x_2 \in A_2 \mid \delta(x_2) \in A_2\}.$$

Clearly, $A_1 = S_{11} \cup S_{12}$, $A_2 = S_{21} \cup S_{22}$, $\delta(S_{11}) \subseteq A_1$, $\delta(S_{12}) \subseteq Z(A) \cap A_2$, $S_{11} \cap S_{12} = \{0\}$, $\delta(S_{21}) \subseteq Z(A) \cap A_1$, $\delta(S_{22}) \subseteq A_2$ and $S_{21} \cap S_{22} = \{0\}$. For any $x = x_1 + x_2 \in A$ where $x_1 \in A_1$ and $x_2 \in A_2$ we define δ_{11}, δ_{12}, δ_{21} and δ_{22} as follows:

$$\delta_{11}(x) = \delta(x_1) \text{ if } x = x_1 \in S_{11},$$

$$\delta_{12}(x) = \delta(x_1) \text{ if } x = x_1 \in S_{12},$$

$$\delta_{21}(x) = \delta(x_2) \text{ if } x = x_2 \in S_{21},$$

$$\delta_{22}(x) = \delta(x_2) \text{ if } x = x_2 \in S_{22},$$

and $\delta_{ij}(x) = 0$ otherwise, for i, j = 1,2. Then we have that $\delta_{11}, \delta_{12}, \delta_{21}, \delta_{22} \in \text{Der}(A)$. In particular, $\delta_{11} \in \text{Der}(A_1)$, $\delta_{12} \in I_1$, $\delta_{21} \in I_2$ and $\delta_{22} \in \text{Der}(A_2)$. By definition, any $\delta \in \text{Der}(A)$ can be written as $\delta = \delta_{11} + \delta_{12} + \delta_{21} + \delta_{22}$. Hence we have $\text{Der}(A) \subseteq (\text{Der}(A_1) + I_1) \oplus (\text{Der}(A_2) + I_2)$. Since the reverse inclusion is clear, we have $\text{Der}(A) = (\text{Der}(A_1) + I_1) \oplus (\text{Der}(A_2) + I_2)$. □

Corollary 3.18. Let the Leibniz algebra $\mathrm{A} = \mathrm{A}_1 \oplus \mathrm{A}_2$ where A_1 and A_2 are ideals of A. Then

(i) if $Z(\mathrm{A}) = \{0\}$, then $\mathrm{Der}(\mathrm{A}) = \mathrm{Der}(\mathrm{A}_1) \oplus \mathrm{Der}(\mathrm{A}_2)$,

(ii) if $\mathrm{A}^2_i = \mathrm{A}_i$ for all $i = 1,2$, then $\mathrm{Der}(\mathrm{A}) = \mathrm{Der}(\mathrm{A}_1) \oplus \mathrm{Der}(\mathrm{A}_2)$,

(iii) if $Z(\mathrm{A}) \cap \mathrm{A}_i \neq \{0\}$ and $\mathrm{A}_j^2 \neq \mathrm{A}_j$ for $i \neq j$, then $\mathrm{Der}(\mathrm{A}) \neq \mathrm{Der}(\mathrm{A}_1) \oplus \mathrm{Der}(\mathrm{A}_2)$.

Proof. (i) Assume that $Z(\mathrm{A}) = \{0\}$. Then we have $\{\delta \in \mathrm{Der}(\mathrm{A}) \mid \delta(\mathrm{A}_2) = \{0\}$ and $\delta(\mathrm{A}_1) = \{0\}\}$ and $\{\delta \in \mathrm{Der}(\mathrm{A}) \mid \delta(\mathrm{A}_1) = \{0\}$ and $\delta(\mathrm{A}_2) = \{0\}\}$. Hence $I_1 = \{0\} = I_2$ which implies $\mathrm{Der}(\mathrm{A}) = \mathrm{Der}(\mathrm{A}_1) \oplus \mathrm{Der}(\mathrm{A}_2)$.

(ii) If $\mathrm{A}_i^2 = \mathrm{A}_i$ for all $i = 1, 2$, then for $\delta \in \mathrm{Der}(\mathrm{A})$ we have $\delta(\mathrm{A}_i) = \delta(\mathrm{A}_i^2) = \delta([\mathrm{A}_i, \mathrm{A}_i]) = [\delta(\mathrm{A}_i), \mathrm{A}_i] + [\mathrm{A}_i, \delta(\mathrm{A}_i)] \subseteq \mathrm{A}_i + \mathrm{A}_i \subseteq \mathrm{A}_i$ for $i = 1, 2$. This implies that $I_i = \{0\}$ for $i = 1, 2$. Hence, $\mathrm{Der}(\mathrm{A}) = \mathrm{Der}(\mathrm{A}_1) \oplus \mathrm{Der}(\mathrm{A}_2)$.

(iii) Assume that $Z(\mathrm{A}) \cap \mathrm{A}_1 \neq \{0\}$ and $\mathrm{A}_2^2 \neq \mathrm{A}_2$. Then there exist $0 \neq x_1 \in Z(\mathrm{A}) \cap \mathrm{A}_1$ and $0 \neq x_2 \in \mathrm{A}_2 \setminus \mathrm{A}_2^2$. Suppose that $x_2 \in \mathrm{A}^2 = \mathrm{A}_1^2 \oplus \mathrm{A}_2^2$. Then $x_2 \in \mathrm{A}_1^2 \cap \mathrm{A}_2 \subseteq \mathrm{A}_1 \cap \mathrm{A}_2 = \{0\}$. Thus, $x_2 = 0$ which is a contradiction. Hence $x_2 \notin \mathrm{A}^2$. Define $\delta\colon \mathrm{A} \to \mathrm{A}$ by $\delta(x_2) = x_1$ and $\delta(x) = 0$ for all $x \neq x_2$. Clearly, for any $x, y \in \mathrm{A}$ we have $[x,y] \neq x_2$ and hence $\delta[x,y] = 0$. Consider

$$x = x_2, y = x_2,\ [\delta(x),y] + [x, \delta(y)] = [\delta(x_2),x_2] + [x_2, \delta(x_2)] = [x_1,x_2] + [x_2,x_1] = 0,$$

$$x = x_2,\ y \neq x_2,\ [\delta(x),y] + [x, \delta(y)] = [\delta(x_2),y] + [x_2, \delta(y)] = [x_1,y] + [x_2,0] = 0,$$

$$x \neq x_2,\ y = x_2,\ [\delta(x),y] + [x, \delta(y)] = [\delta(x),x_2] + [x, \delta(x_2)] = [0,x_2] + [x,x_1] = 0,$$

$$x \neq x_2,\ y \neq x_2,\ [\delta(x),y] + [x, \delta(y)] = [\delta(x),y] + [x, \delta(y)] = [0,y] + [x,0] = 0.$$

This implies $\delta[x,y] = [\delta(x),y] + [x,\delta(y)]$ for all $x, y \in \mathrm{A}$. Thus, δ is a derivation of A. Since $\delta(\mathrm{A}_1) = \{0\}$ and $\delta(\mathrm{A}_2) = \mathrm{span}\{x_1\} \subseteq Z(\mathrm{A}) \cap \mathrm{A}_1$, $\delta \in I_2$ which implies $\emptyset \neq I_2 \neq \{0\}$. Hence $\mathrm{Der}(\mathrm{A}) \neq \mathrm{Der}(\mathrm{A}_1) \oplus \mathrm{Der}(\mathrm{A}_2)$. □

Example 3.19. Consider the Leibniz algebra $\mathrm{A} = \mathrm{A}_1 \oplus \mathrm{A}_2$ where $\mathrm{A}_1 = \mathrm{span}\{x, y, z\}$ and $\mathrm{A}_2 = \mathrm{span}\{a, b, c\}$ with the non-zero multiplications in A given by $[x,z] = \alpha z$, $\alpha \in \mathbb{F} \setminus \{0\}$, $[x,y] = y$, $[y,x] = -y$, $[a,a] = c$, $[a,b] = b$ and $[b,a] = -b$. By direct calculations, we have that $\mathrm{Der}(\mathrm{A}) = \mathrm{span}\{\delta_1, \delta_2, \delta_3, \delta_4, \delta_5, \delta_6, \delta_7\}$ where

$\delta_1(x) = y,\quad \delta_1(y) = 0,\quad \delta_1(z) = 0,\quad \delta_1(a) = 0,\quad \delta_1(b) = 0,\quad \delta_1(c) = 0,$

$\delta_2(x) = 0,\quad \delta_2(y) = y,\quad \delta_2(z) = 0,\quad \delta_2(a) = 0,\quad \delta_2(b) = 0,\quad \delta_2(c) = 0,$

$\delta_3(x) = 0$, $\delta_3(y) = 0$, $\delta_3(z) = z$, $\delta_3(a) = 0$, $\delta_3(b) = 0$, $\delta_3(c) = 0$,

$\delta_4(x) = 0$, $\delta_4(y) = 0$, $\delta_4(z) = 0$, $\delta_4(a) = b$, $\delta_4(b) = 0$, $\delta_4(c) = 0$,

$\delta_5(x) = 0$, $\delta_5(y) = 0$, $\delta_5(z) = 0$, $\delta_5(a) = c$, $\delta_5(b) = 0$, $\delta_5(c) = 0$,

$\delta_6(x) = 0$, $\delta_6(y) = 0$, $\delta_6(z) = 0$, $\delta_6(a) = 0$, $\delta_6(b) = b$, $\delta_6(c) = 0$,

$\delta_7(x) = c$, $\delta_7(y) = 0$, $\delta_7(z) = 0$, $\delta_7(a) = 0$, $\delta_7(b) = 0$, $\delta_7(c) = 0$.

In this case, $Z(\mathsf{A}) = \text{span}\{c\}$. Hence, $\text{Der}(\mathsf{A}_1) = \text{span}\{\delta_1, \delta_2, \delta_3\}$, $\text{Der}(\mathsf{A}_2) = \text{span}\{\delta_4, \delta_5, \delta_6\}$ and $\delta_7 \in I_1$. Note that and $Z(\mathsf{A}) \cap \mathsf{A}_2 = \text{span}\{c\} \neq \{0\}$ and $\mathsf{A}_1^2 = \text{span}\{y, z\} \neq \mathsf{A}_1$.

Example 3.20. Consider the Leibniz algebra $\mathsf{A} = \mathsf{A}_1 \oplus \mathsf{A}_2$ where $\mathsf{A}_1 = \text{span}\{x, y, z\}$ and $\mathsf{A}_2 = \text{span}\{a, b, c\}$ with non-zero brackets defined by $[x,z] = 2z$, $\alpha \in \mathbb{F}$, $[y,y] = z$, $[x,y] = y$, $[y,x] = -y$, $[a,c] = \alpha c$, $\alpha \in \mathbb{F}$, $[a,b] = b$ and $[b,a] = -b$. Observe that $Z(\mathsf{A}) = \{0\}$. By direct calculations, we can see that $\text{Der}(\mathsf{A}) = \text{span}\{\delta_1, \delta_2, \delta_3, \delta_4, \delta_5\}$ where

$\delta_1(x) = y$, $\delta_1(y) = -z$, $\delta_1(z) = 0$, $\delta_1(a) = 0$, $\delta_1(b) = 0$, $\delta_1(c) = 0$,

$\delta_2(x) = z$, $\delta_2(y) = y$, $\delta_2(z) = 2z$, $\delta_2(a) = 0$, $\delta_2(b) = 0$, $\delta_2(c) = 0$,

$\delta_3(x) = 0$, $\delta_3(y) = 0$, $\delta_3(z) = 0$, $\delta_3(a) = b$, $\delta_3(b) = 0$, $\delta_3(c) = 0$,

$\delta_4(x) = 0$, $\delta_4(y) = 0$, $\delta_4(z) = 0$, $\delta_4(a) = 0$, $\delta_4(b) = b$, $\delta_4(c) = 0$,

$\delta_5(x) = 0$, $\delta_5(y) = 0$, $\delta_5(z) = 0$, $\delta_5(a) = 0$, $\delta_5(b) = 0$, $\delta_5(c) = c$.

In this case, $\text{Der}(\mathsf{A}_1) = \text{span}\{\delta_1, \delta_2\}$ and $\text{Der}(\mathsf{A}_2) = \text{span}\{\delta_3, \delta_4, \delta_5\}$. Thus, $\text{Der}(\mathsf{A}) = \text{Der}(\mathsf{A}_1) \oplus \text{Der}(\mathsf{A}_2)$.

In (6), Meng proved that for a Lie algebra $\mathsf{L} = \mathsf{L}_1 \oplus \mathsf{L}_2$ where L_1 and L_2 are ideals of L and for any subspace M of L such that $\mathsf{L}_1 \subseteq M$, $M = \mathsf{L}_1 \oplus (\mathsf{L}_2 \cap M)$ and M is an ideal of L if and only if $\mathsf{L}_2 \cap M$ is an ideal of L_2. The following lemma is the analog for Leibniz algebras.

Lemma 3.21. Let the Leibniz algebra $\mathsf{A} = \mathsf{A}_1 \oplus \mathsf{A}_2$ where A_1 and A_2 are ideals of A. Let M be a subalgebra of A and $\mathsf{A}_1 \subseteq M$. Then

$$M = \mathsf{A}_1 \oplus (\mathsf{A}_2 \cap M)$$

and M is an ideal of A if and only if $\mathsf{A}_2 \cap M$ is an ideal of A_2.

Proof. It is clear that $M = \mathsf{A} \cap M = (\mathsf{A}_1 \oplus \mathsf{A}_2) \cap M = \mathsf{A}_1 \oplus (\mathsf{A}_2 \cap M)$ because $\mathsf{A}_1 \subseteq M$. Suppose that M is an ideal of A. Then $\mathsf{A}_2 \cap M$ is an ideal of A, hence an ideal of A_2. Conversely, assume that $\mathsf{A}_2 \cap M$ is an ideal of A_2. To show M is an ideal of A, let $a \in \mathsf{A}$ and $h \in M = \mathsf{A}_1 \oplus (\mathsf{A}_2 \cap M)$. Then there exist $a_1 \in \mathsf{A}_1$, $a_2 \in \mathsf{A}_2$, $b_1 \in \mathsf{A}_1$ and $b_2 \in \mathsf{A}_2 \cap M$ such that $a = a_1 + a_2$ and $h = b_1 + b_2$. Then we have $[a,h] = [a, b_1 + b_2] = [a,b_1] + [a_1,b_2] + [a_2,b_2]$ and $[h,a] = [b_1,a] + [b_2,a_1] + [b_2,a_2]$. Since A_1 is an ideal of A, $[a,b_1]$, $[a_1,b_2]$, $[b_1,a]$, $[b_2,a_1]$ $\in \mathsf{A}_1 \subseteq M$. Since $\mathsf{A}_2 \cap M$ is an ideal of A_2, $[a_2,b_2]$, $[b_2,a_2] \in \mathsf{A}_2 \cap M \subseteq M$. Therefore, $[a,h]$, $[h,a] \in M$ which implies M is an ideal of A. □

In (10), Tôgô studied the properties of inner derivations of Lie algebras by comparing them with the set of central derivations. Here, we delve into similar findings for left Leibniz algebras. Note that Shermatova and Khudoyberdiyev, in (19), also studied central derivations by comparing them with inner derivations. However, their focus was on right Leibniz algebras, employing the definition of inner derivations provided in (8).

Definition 3.22. Let A be a Leibniz algebra. A derivation $d \in \mathrm{Der}(\mathsf{A})$ is called a *central derivation if* $\mathrm{im}(d) \subseteq Z(\mathsf{A})$.

We denote CDer(A) to be the set of all central derivations of A. It is easy to see that CDer(A) is a subalgebra of Der(A). We start by examining derivations of Leibniz algebras that are both inner and central. For a Leibniz algebra A, by Theorem 3.7, we have that $\mathrm{IDer}(\mathsf{A}) = L(\mathsf{A}) + I$ where $I = \{d \in Der(\mathsf{A}) \mid \mathrm{im}(d) \subseteq \mathrm{Leib}(\mathsf{A})\}$. The following proposition is the Leibniz algebra analogue of the result in [(10), Lemma 2].

Proposition 3.23. (18) Let A be a Leibniz algebra and J = I ∩ CDer(A). Then

(i) IDer(A) ∩ CDer(A) = $L(Z_1) + J$ where $Z_1 = \{x \in A \mid [x, A] \subseteq Z(A)\}$,

(ii) IDer(A) ∩ CDer(A) ⊆ $L(Z_2) + J$ where $Z_2 = \{r \in \mathrm{rad}(A) \mid [r, \mathrm{rad}(A^2)] = 0\}$.

Proof. (i) IDer(A) ∩ CDer(A) = $L(A)$ ∩ CDer(A) + I ∩ CDer(A) = $\{L_x \mid \mathrm{im}(L_x) \subseteq Z(A)\} + J = L(Z_1) + J$ where $Z_1 = \{x \in A \mid [x,A] \subseteq Z(A)\}$.

(ii) Let $d \in$ IDer(A) ∩ CDer(A). By (i), there exist $z \in Z_1$ and $h \in J$ such that $d = L_z + h$. By Theorem 2.18, there exists a semisimple Lie algebra S such that A = S + rad(A) and S ∩ rad(A) = {0}. Thus, $A^2 = S + \mathrm{rad}(A^2)$ and there exist $s \in S$ and $r \in \mathrm{rad}(A)$ such that $z = s + r$. Since $\mathrm{im}(h) \subseteq Z(A)$, we have $h(S) = h([S,S]) = 0$ and hence $h(\mathrm{rad}(A^2)) = h(A^2) = 0$. Since $\mathrm{im}(d) \subseteq Z(A)$, we also have $d(S) = 0$ and $d(A^2) = 0$ which implies that $d(\mathrm{rad}(A^2)) = 0$. It follows that $0 = L_{s+r}(S) = [s + r, S] = [s,S] + [r,S]$. Hence, $[s,S] = 0$, and so $s = 0$. Therefore, $d = L_r + h$ and $[r, \mathrm{rad}(A^2)] = 0$. □

Example 3.24. Consider the Leibniz algebra A = span{w, x, y, z} with non-zero brackets defined by $[w,x] = y$, $[x,w] = z$, $[w,y] = z$ and $[x,x] = z$. Clearly Leib(A) = span{y, z} and $Z(A)$ = span{z}. By direct calculations, we have that Der(A) = span{d_1, d_2, d_3} = IDer(A) = I where

$d_1(w) = z$,	$d_1(x) = 0$,	$d_1(y) = 0$,	$d_1(z) = 0$,
$d_2(w) = 0$,	$d_2(x) = y$,	$d_2(y) = z$,	$d_2(z) = 0$,
$d_3(w) = 0$,	$d_3(x) = z$,	$d_3(y) = 0$,	$d_3(z) = 0$.

Then CDer(A) = span{d_1, d_3} = J and Z_1 = span{x, y, z}. Then IDer(A) ∩ CDer(A) = $L(Z_1) + J$. Moreover, we can see that A = rad(A), rad(A^2) = span{y, z} and Z_2 = span{x, y, z}. Therefore, IDer(A) ∩ CDer(A) ⊆ $L(Z_2) + J$.

Following this, we delve into Leibniz algebras in which all central derivations are inner, resulting in the Leibniz algebra analogue of [(10), Lemma 3].

Theorem 3.25. (18) Let A be a Leibniz algebra satisfying CDer(A) $\subseteq$ IDer(A). If rad(A) is abelian, then either Z(A) = {0} or A = A^2.

Proof. Let A be a Leibniz algebra satisfying CDer(A) $\subseteq$ IDer(A). By Theorem 2.18, there exists a semisimple Lie algebra S such that A = S + rad(A) and $S \cap$ rad(A) = {0}. Suppose that Z(A) $\neq$ {0} and A $\neq$ A^2. Since $A^2 = S + [S,$ rad(A)] + [rad(A), S], we have [S, rad(A)] + [rad(A), S] $\subsetneq$ rad(A). Choose a subspace U of rad(A) such that rad(A) = U + [S, rad(A)] + [rad(A), S] and $U \cap$ ([S, rad(A)] + [rad(A), S]) = {0}. Define a nonzero linear map d : A $\rightarrow$ A such that $d(U) \subseteq Z$(A) and $d(S$ + [S, rad(A)] + [rad(A), S]) = 0. Clearly, d is a central derivation of A. Since CDer(A) $\subseteq$ IDer(A) = L(A) + I and A = S + rad(A), there exist $s \in S$, $r \in$ rad(A) and $h \in I$ such that $d = L_{s+r} + h$. Since $d(S) = 0$ and $[r, S] + h(S) \subseteq$ rad(A), we have $[s,S] = 0$, and hence $s = 0$. This implies that $d(U) = [r,U] + h(U) \subseteq$ Leib(A) since $[r,U] \subseteq$ [rad(A), rad(A)] = {0}. Let $0 \neq u \in U$. Then $d(u) = \alpha[x,x]$ for some $\alpha \in \mathbb{F}$ and $x \in$ A. Since S is a subalgebra, $x \notin S$ which implies $x \in$ rad(A). Hence, $d(u) = \alpha[x,x] \in$ [rad(A), rad(A)] = {0} which contradicts our definition of d. Therefore, we have either Z(A) = {0} or A = A^2. □

Corollary 3.26. Let A be a Leibniz algebra satisfying CDer(A) $\subseteq$ IDer(A). If Z(A) $\neq${0} and CDer(A) $\neq${0}, then rad(A) is not abelian.

Proof. Let A be a Leibniz algebra such that CDer(A) $\subseteq$ IDer(A). Suppose that Z(A) $\neq${0} and CDer(A) $\neq${0}. If rad(A) is abelian, then by Theorem 6.4, A = A^2. Hence for all $d \in$ CDer(A), d(A) = d([A,A]) = {0} which implies that d = 0. It follows that CDer(A) = {0}, a contradiction. Hence, rad(A) is not abelian. □

Example 3.27. Consider the Leibniz algebra A = span{x, y, z} with non-zero multiplications defined by $[x,y] = y$, $[y,x] = -y$ and $[x,x] = z$. From Example 3.9, we have that Der(A) = span{d_1, d_2, d_3} = IDer(A) where

$$d_1(x) = y, \quad d_1(y) = 0, \quad d_1(z) = 0,$$

$$d_2(x) = z, \quad d_2(y) = 0, \quad d_2(z) = 0,$$

$$d_3(x) = 0, \quad d_3(y) = y, \quad d_3(z) = 0.$$

It is easy to see that $Z(\mathsf{A}) = \mathrm{span}\{z\} \neq \{0\}$ and $\mathrm{CDer}(\mathsf{A}) = \mathrm{span}\{d_2, d_3\} \subseteq \mathrm{IDer}(\mathsf{A})$. In this case, $\mathrm{rad}(\mathsf{A}) = \mathsf{A}$ because $\mathsf{A}^{(3)} = \{0\}$. Hence, $\mathrm{rad}(\mathsf{A})$ is not abelian.

To conclude, we investigate Leibniz algebras in which all inner derivations are central, thereby establishing the Leibniz algebra analogue of [(10), Theorem 3].

Theorem 3.28. (18) Let A be a Leibniz algebra. Then

(i) $\mathrm{IDer}(\mathsf{A}) \subseteq \mathrm{CDer}(\mathsf{A})$ if and only if $\mathsf{A}^2 \subseteq Z(\mathsf{A})$ if and only if $\mathsf{A}^3 = \{0\}$,

(ii) If $Z(\mathsf{A}) \neq \{0\}$ and $\mathrm{IDer}(\mathsf{A}) = \mathrm{CDer}(\mathsf{A})$, then $\mathsf{A}^2 = Z(\mathsf{A})$.

Proof. (i) Suppose that $\mathrm{IDer}(\mathsf{A}) \subseteq \mathrm{CDer}(\mathsf{A})$. Then for all $x, y \in \mathsf{A}$, $L_x \in \mathrm{IDer}(\mathsf{A}) \subseteq \mathrm{CDer}(\mathsf{A})$ and $[x,y] = L_x(y) \in Z(\mathsf{A})$. Conversely, assume that $\mathsf{A}^2 \subseteq Z(\mathsf{A})$. If $d \in \mathrm{IDer}(\mathsf{A})$, then there exists $a \in \mathsf{A}$ such that $d(x) - L_a(x) \in \mathrm{Leib}(\mathsf{A})$ for any $x \in \mathsf{A}$ which implies that $d(x) \in \mathsf{A}^2 \subseteq Z(\mathsf{A})$ and $d \in \mathrm{CDer}(\mathsf{A})$. Hence, $\mathrm{IDer}(\mathsf{A}) \subseteq \mathrm{CDer}(\mathsf{A})$. Clearly, $\mathsf{A}^2 \subseteq Z(\mathsf{A})$ if and only if $\mathsf{A}^3 = [\mathsf{A},[\mathsf{A},\mathsf{A}]] = 0$.

(ii) Suppose that $Z(\mathsf{A}) \neq \{0\}$ and $\mathrm{IDer}(\mathsf{A}) = \mathrm{CDer}(\mathsf{A})$. By (i), $\mathsf{A}^2 \subseteq Z(\mathsf{A})$. If $\mathsf{A}^2 \neq Z(\mathsf{A})$, then by [(20), Theorem 3.6], A has an outer central derivation which contradicts our assumption. Thus, $\mathsf{A}^2 = Z(\mathsf{A})$. □

Note that [(10), Theorem 3 (iii)] is also valid in our case. In [(10), Theorem 3 (ii)], Tôgô proved that for a Lie algebra L, if $Z(\mathsf{L}) \neq 0$, then $\mathrm{IDer}(\mathsf{L}) = \mathrm{CDer}(\mathsf{L})$ if and only if $\mathsf{L}^2 = Z(\mathsf{L})$ and $\dim(Z(\mathsf{L})) = 1$. However, as the following example shows, there exists a Leibniz algebra A where $Z(\mathsf{A}) \neq \{0\}$ and $\mathrm{IDer}(\mathsf{A}) = \mathrm{CDer}(\mathsf{A})$, but $\dim(Z(\mathsf{A})) > 1$.

Example 3.29. Consider the Leibniz algebra $\mathsf{A} = \mathrm{span}\{w, x, y, z\}$ with non-zero brackets defined by $[w,w] = z$, $[w,x] = y$ and $[x,w] = -y$. We can see that $Z(\mathsf{A}) = \mathsf{A}^2 = \mathrm{span}\{y, z\}$, $\mathrm{Leib}(\mathsf{A}) = \mathrm{span}\{z\}$ and $\mathrm{Der}(\mathsf{A}) = \mathrm{span}\{d_1, d_2, d_3, d_4, d_5, d_6, d_7\}$ where

$d_1(w) = w,$	$d_1(x) = 0,$	$d_1(y) = y,$	$d_1(z) = 2z,$
$d_2(w) = 0,$	$d_2(x) = x,$	$d_2(y) = y,$	$d_2(z) = 0,$

$d_3(w) = x$,	$d_3(x) = 0$,	$d_3(y) = 0$,	$d_3(z) = 0$,
$d_4(w) = y$,	$d_4(x) = 0$,	$d_4(y) = 0$,	$d_4(z) = 0$,
$d_5(w) = z$,	$d_5(x) = 0$,	$d_5(y) = 0$,	$d_5(z) = 0$,
$d_6(w) = 0$,	$d_6(x) = y$,	$d_6(y) = 0$,	$d_6(z) = 0$,
$d_7(w) = 0$,	$d_7(x) = z$,	$d_7(y) = 0$,	$d_7(z) = 0$.

Then IDer(**A**) = span$\{d_4, d_5, d_6, d_7\}$ = CDer(**A**).

CHAPTER 4
COMPLETE LEIBNIZ ALGEBRAS

A Lie algebra **L** is said to be complete if its center is trivial and all derivations are inner, i.e., for each derivation δ on **L**, there exists $x \in$ **L** such that $\delta = \mathrm{ad}_x$. Otherwise, the derivation is called outer. In 2013, Ancochea and Campoamor (8) gave a definition of complete Leibniz algebras analogous to complete Lie algebras, i.e., a Leibniz algebra **A** is said to be complete if Z(**A**) = {0} and for each derivation δ on **A**, there exists $x \in$ **A** such that $\delta = L_x$. However, the signature properties of a complete Lie algebra did not extend to complete Leibniz algebras under this definition. Motivated by this, in 2020, Boyle, Misra, and Stitzinger (9) defined a complete Leibniz algebra as follows.

Definition 4.1. (9) A Leibniz algebra **A** is *complete* if

(i) Z(**A** / Leib(**A**)) = {0} and

(ii) all derivations of **A** are inner, i.e., Der(**A**) = IDer(**A**).

Let **A** be a Lie algebra and hence a Leibniz algebra. If **A** is complete as a Leibniz algebra, then it is complete as a Lie algebra because Leib(**A**) = {0} and for each derivation δ on **A**, there exists $x \in$ **A** such that $\delta = L_x = \mathrm{ad}_x$. Throughout this work, we will refer to complete Leibniz algebras of Leibniz algebras that satisfy Definition 4.1.

Example 4.2. (12) Consider the Leibniz algebra **A** = span{x, y, z} with non-zero brackets defined by $[x,z] = z$. We can see that Z(**A**) = span{y} and Leib(**A**) = span{z}. Hence Z(**A** / Leib(**A**)) = **A** / Leib(**A**) which is not trivial. Then **A** is not complete.

Example 4.3. (12) Consider the Leibniz algebra **A** = span{x, y, z} with non-zero brackets defined by $[x,z] = \alpha z$, $[x,y] = y$ and $[y,x] = -y$ for some $\alpha \in \mathbb{F} \setminus \{0\}$. We can see that Z(**A**) = {0} and Leib(**A**) = span{z}. Then Z(**A** / Leib(**A**)) = {0} and Der(**A**) = span{$\delta_1, \delta_2, \delta_3$} where

$$\delta_1(x) = y,\ \delta_1(y) = 0,\ \delta_1(z) = 0,$$

$$\delta_2(x) = 0,\ \delta_2(y) = y,\ \delta_2(z) = 0,$$

$$\delta_3(x) = 0,\ \delta_3(y) = 0,\ \delta_3(z) = z.$$

Since $\text{im}(\delta_1 - L_{-y}) \subseteq \text{Leib}(\mathbf{A})$, $\text{im}(\delta_2 - L_x) \subseteq \text{Leib}(\mathbf{A})$ and $\text{im}(\delta_3 - L_0) \subseteq \text{Leib}(\mathbf{A})$, we have that $\mathbf{A}$ is complete.

Example 4.4. (12) Consider the Leibniz algebra $\mathbf{A} = \text{span}\{x, y, z\}$ with non-zero brackets defined by $[x,x] = z, [x,y] = y$ and $[y,x] = -y$. We can see that $Z(\mathbf{A}) = \text{span}\{z\} = \text{Leib}(\mathbf{A})$, $Z(\mathbf{A} / \text{Leib}(\mathbf{A})) = \{0\}$ and $\text{Der}(\mathbf{A}) = \text{span}\{\delta_1, \delta_2, \delta_3\}$ where

$$\delta_1(x) = y, \quad \delta_1(y) = 0, \quad \delta_1(z) = 0,$$

$$\delta_2(x) = z, \quad \delta_2(y) = 0, \quad \delta_2(z) = 0,$$

$$\delta_3(x) = 0, \quad \delta_3(y) = y, \quad \delta_3(z) = 0.$$

Since $\text{im}(\delta_1 - L_{-y}) \subseteq \text{Leib}(\mathbf{A})$, $\text{im}(\delta_2 - L_z) \subseteq \text{Leib}(\mathbf{A})$ and $\text{im}(\delta_3 - L_x) \subseteq \text{Leib}(\mathbf{A})$, we have that $\mathbf{A}$ is complete.

Example 4.5. (12) Consider the Leibniz algebra $\mathbf{A} = \text{span}\{x, y, z\}$ with non-zero brackets defined by $[x,z] = 2z$, $[y,y] = y$, $[x,x] = z$, $[x,y] = y$ and $[y,x] = -y$. We can see that $Z(\mathbf{A}) = \{0\}$ and $\text{Leib}(\mathbf{A}) = \text{span}\{z\}$. Then $Z(\mathbf{A} / \text{Leib}(\mathbf{A})) = \{0\}$ and $\text{Der}(\mathbf{A}) = \text{span}\{\delta_1, \delta_2\}$ where

$$\delta_1(x) = y, \quad \delta_1(y) = -z, \quad \delta_1(z) = 0,$$

$$\delta_2(x) = z, \quad \delta_2(y) = y, \quad \delta_2(z) = 2z.$$

Since $\text{im}(\delta_1 - L_{-y}) \subseteq \text{Leib}(\mathbf{A})$ and $\text{im}(\delta_2 - L_x) \subseteq \text{Leib}(\mathbf{A})$, we have that $\mathbf{A}$ is complete.

Remark 4.6. It should be noted that the Leibniz algebra in Example 4.5 is complete, whereas Examples 4.2, 4.3, and 4.4 are not complete in the sense of (8). This is because the centers of Examples 4.2 and 4.4 are not trivial, and there exists an outer derivation in Example 4.3 by the definition in (8).

It is known that semisimple Lie algebras are complete. In (9), Boyle, Misra, and Stitzinger proved that semisimple Leibniz algebras are also complete using Definition 4.1. The following example shows that there exists a semisimple Leibniz algebra which is not complete in the sense of (8).

Example 4.7. (9) Let $S = sl(2,\mathbb{C})$ and $V = \mathbb{C}^2$. It is known that V is an irreducible S module under the matrix multiplication. Define $\mathsf{A} = S \oplus V$ with brackets in A given by $[x,y] = xy - yx = -[y,x]$, $[x,u] = xu$, $[u,x] = 0 = [v,u]$ for all $x, y \in S$, $u, v \in V$. Then A is a simple, hence semisimple Leibniz algebra with Leib(A) = V. Define the linear operator $T : \mathsf{A} \times \mathsf{A} \to \mathsf{A}$ by $T(x) = 0, T(u) = u$ for all $x \in S$, $u \in V$. Then for $x + u, y + v \in \mathsf{A}$, $T[x + u, y + v] = T(xy + xv) = xv$, $[T(x + u), y + v] = 0$ and $[x + u, T(y + v)] = [x + u, v] = xv$. Hence T is a derivation. If $T = L_{x+u}$ for some $x + u \in \mathsf{A}$, then for all $y + v \in \mathsf{A}$, $v \neq 0$ we have $v = T(y + v) = [x + u, y + v] = [x,y] + xv$ which implies $[x,y] = 0$, $xv = v$ for all $y \in S$, $v \in V$. This implies $x \in Z(S) = \{0\}$ which is a contradiction since $v = xv = 0v = 0$. Therefore, T is not inner and A is not complete by the definition in (8).

It is known that non-zero nilpotent Lie algebras are not complete (6). In (9), Boyle, Misra, and Stitzinger proved that the statement also holds for non-zero nilpotent Leibniz algebras. It is also known that a nilpotent Lie algebra contains outer derivations (7). However, the example below shows that there exist nilpotent Leibniz algebras that do not have outer derivations.

Example 4.8. (9) Consider the Leibniz algebra A = span$\{w, x, y, z\}$ with non-zero brackets $[x,x] = z$, $[w,x] = [x,w] = -y + z$, $[w,y] = -z$. Clearly, Leib(A) = span$\{z\} \subseteq \mathsf{A}^2$ = span$\{y, z\}$ and $\mathsf{A}^4 = \{0\}$. So A is nilpotent. By direct calculations, we have that Der(A) = span$\{\delta_1, \delta_2, \delta_3, \delta_4\}$ where

$\delta_1(w) = y$,	$\delta_1(x) = 0$,	$\delta_1(y) = 0$,	$\delta_1(z) = 0$,
$\delta_2(w) = z$,	$\delta_2(x) = 0$,	$\delta_2(y) = 0$,	$\delta_2(z) = 0$,
$\delta_3(w) = 0$,	$\delta_3(x) = y$,	$\delta_3(y) = z$,	$\delta_3(z) = 0$,
$\delta_4(w) = 0$,	$\delta_4(x) = z$,	$\delta_4(y) = 0$,	$\delta_4(z) = 0$.

Note that $L_w = \delta_3$, $L_x = -\delta_1 + \delta_2 + \delta_4$ and $L_y = -\delta_2$. By definition im(δ_i) $\subseteq$ Leib(A) for $i = 2, 4$. Also im($\delta_3 - L_w$) $\subseteq$ Leib(A) and im($\delta_1 + L_x$) = im($\delta_2 + \delta_4$) $\subseteq$ Leib(A). Hence by linearity all derivations of A are inner.

Consider the Leibniz algebra A_n = span$\{e_1,e_2,...,e_n,e\}$ with non-zero brackets $[e_1,e_i] = e_{i+1}$,$[e_1,e] = e_1$ and $[e,e_i] = -ie_i$ for $i = 1,...,n$. In (8), it is proved that Der(A_n) = $L(A_n)$, A_n is solvable and complete for all $n \geqslant 1$ by the definition of completeness in (8). Here we show that this solvable Leibniz algebra A_n remains complete under Definition 4.1.

Proposition 4.9. The Leibniz algebra A_n is complete for all $n \geqslant 1$.

Proof. Let $n \geqslant 1$. Since Der(A_n) = $L(A_n)$, all derivations of A_n are inner. For all $x = \alpha_1 e_1 + \alpha_2 e_2 + \cdots + \alpha_n e_n + \alpha e \in A_n$, we have that

$$
\begin{aligned}
[x,x] &= [\alpha_1 e_1 + \alpha_2 e_2 + \cdots + \alpha_n e_n + \alpha e,\ \alpha_1 e_1 + \alpha_2 e_2 + \cdots + \alpha_n e_n + \alpha e] \\
&= [\alpha_1 e_1, \textstyle\sum_{i=1}^{n} \alpha_i e_i + \alpha e] + [\alpha e, \textstyle\sum_{i=1}^{n} \alpha_i e_i] \\
&= (\textstyle\sum_{i=1}^{n-1} \alpha_1 \alpha_i e_{i+1} + \alpha_1 \alpha e) - \textstyle\sum_{i=1}^{n} \alpha\alpha_i i e_i \\
&= \textstyle\sum_{i=1}^{n-1} \alpha_1 \alpha_i e_{i+1} - \textstyle\sum_{i=2}^{n} \alpha\alpha_i i e_i.
\end{aligned}
$$

Thus, Leib(A_n) = span$\{e_2, e_3,..., e_n\}$ and hence A_n / Leib(A_n) = span$\{e_1$ + Leib(A_n), e + Leib(A_n)$\}$. It is easy to see that $Z(A_n$ / Leib(A_n)) = $\{0\}$. Therefore, A_n is complete. □

Proposition 4.10. Let A be a Leibniz algebra and A^2 = Leib(A). Then A is complete if and only if A = A^2.

Proof. If A = A^2, then A = Leib(A). It is easy to see that Z(A / Leib(A)) = $\{0\}$. Clearly, for any $\delta \in$ Der(A), im(δ) $\subseteq$ A = Leib(A). Therefore, A is complete. Conversely, if $A^2 \subsetneq A$, then there exists $0 \neq x \in A \setminus A^2$ such that [x + Leib(A), y + Leib(A)] = [x,y] + Leib(A) = Leib(A) for any $y \in$ A because Leib(A) = A^2. This means Leib(A) $\neq x$ + Leib(A) $\in Z$(A / Leib(A)) which implies that A is not complete. □

In (6), Meng proved that for a Lie algebra L = $L_1 \oplus L_2$, then L is complete if and only if L_1 and L_2 are complete. The following theorem is the Leibniz algebra analog of the Lie algebra result.

Theorem 4.11. Let the Leibniz algebra A = $A_1 \oplus A_2$ where A_1 and A_2 are ideals of A. Then A is a complete Leibniz algebra if and only if A_1 and A_2 are complete.

Proof. Assume that A is complete. Then $Z(A / \text{Leib}(A)) = \{0\}$. By Corollary 3.16, for $i = 1, 2$, $Z(A_i / \text{Leib}(A_i)) = \{0\}$. Let $\delta_1 \in \text{Der}(A_1)$ and $\delta_2 \in \text{Der}(A_2)$. Since all derivations of A are inner, for each $i = 1, 2$, there exists $x_i \in A$ such that $\text{im}(\delta_i - L_{x_i}) \subseteq \text{Leib}(A)$. Let $b_1 \in A_1$ and $b_2 \in A_2$. Then $\delta_1(b_1) - L_{x_1}(b_1) \in \text{Leib}(A)$ and $\delta_2(b_2) - L_{x_2}(b_2) \in \text{Leib}(A)$. Thus, $\delta_1(b_1) - L_{x_1}(b_1) + \delta_2(b_2) - L_{x_2}(b_2) \in \text{Leib}(A)$. Since $x_i \in A$, there exist $x_{i1} \in A_1$ and $x_{i2} \in A_2$ such that $x = x_{i1} + x_{i2}$. Note that

$$\begin{aligned}\delta_1(b_1) - L_{x_1}(b_1) + \delta_2(b_2) - L_{x_2}(b_2) &= \delta_1(b_1) - L_{x_{11}+x_{12}}(b_1) + \delta_2(b_2) - L_{x_{21}+x_{22}}(b_2)\\ &= \delta_1(b_1) - L_{x_{11}}(b_1) - L_{x_{12}}(b_1) + \delta_2(b_2) - L_{x_{21}}(b_2) - L_{x_{22}}(b_2)\\ &= \delta_1(b_1) - L_{x_{11}}(b_1) + \delta_2(b_2) - L_{x_{22}}(b_2).\end{aligned}$$

This is because $[A_1, A_2] \subseteq A_1 \cap A_2 = \{0\}$ implies that $L_{x_{12}}(b_1) = 0 = L_{x_{21}}(b_2)$. Thus, we have $\delta_1(b_1) - L_{x_{11}}(b_1) + \delta_2(b_2) - L_{x_{22}}(b_2) \in \text{Leib}(A) = \text{Leib}(A_1) \oplus \text{Leib}(A_2)$. By Theorem 3.15 (ii), $\delta_1(b_1) - L_{x_{11}}(b_1) \in \text{Leib}(A_1)$ and $\delta_1(b_2) - L_{x_{22}}(b_2) \in \text{Leib}(A_2)$. This implies that δ_1 and δ_2 are inner. Hence, A_1 and A_2 are complete. Conversely, assume that A_1 and A_2 are complete. Then $Z(A_i / \text{Leib}(A_i)) = \{0\}$ for $i = 1, 2$. By Corollary 3.16, $Z(A / \text{Leib}(A)) = \{0\}$. Let $\delta \in \text{Der}(A) = (\text{Der}(A_1) + I_1) \oplus (\text{Der}(A_2) + I_2)$. Then $\delta = \delta_{11} + \delta_{12} + \delta_{21} + \delta_{22}$ where $\delta_{11} \in \text{Der}(A_1)$, $\delta_{12} \in I_1$, $\delta_{21} \in I_2$, $\delta_{22} \in \text{Der}(A_2)$. Since δ_{11} and δ_{22} are inner, there exist $x_1 \in A_1$ and $x_2 \in A_2$ such that $\delta_{11}(y_1) - L_{x_1}(y_1) \in \text{Leib}(A_1)$ and $\delta_{22}(y_2) - L_{x_2}(y_2) \in \text{Leib}(A_2)$ for all $y_1 \in A_1$, $y_2 \in A_2$. Since $\delta_{12}(y_1), \delta_{21}(y_2) \in Z(A)$, for all $x \in A$ we have that

$$[\delta_{12}(y_1) + \text{Leib}(A), x + \text{Leib}(A)] = [\delta_{12}(y), x] + \text{Leib}(A) = \text{Leib}(A),$$

$$[\delta_{21}(y_2) + \text{Leib}(A), x + \text{Leib}(A)] = [\delta_{21}(y), x] + \text{Leib}(A) = \text{Leib}(A).$$

This implies $\delta_{12}(y_1) + \text{Leib}(A), \delta_{21}(y_2) + \text{Leib}(A) \in Z(A / \text{Leib}(A)) = \{0\}$ and hence $\delta_{12}(y_1)$, $\delta_{21}(y_2) \in \text{Leib}(A)$. Let $x = x_1 + x_2$. Then, for all $y = y_1 + y_2$ where $y_1 \in A_1$ and $y_2 \in A_2$, we have that

$$\begin{aligned}\delta(y) - L_x(y) &= \delta(y_1 + y_2) - L_{x_1+x_2}(y_1 + y_2)\\ &= \delta(y_1) + \delta(y_2) - L_{x_1}(y_1 + y_2) - L_{x_2}(y_1 + y_2)\\ &= \delta_{11}(y_1) + \delta_{12}(y_1) + \delta_{21}(y_1) + \delta_{22}(y_1) + \delta_{11}(y_2) + \delta_{12}(y_2)\\ &\quad + \delta_{21}(y_2) + \delta_{22}(y_2) - L_{x_1}(y_1) - L_{x_1}(y_2) - L_{x_2}(y_1) - L_{x_2}(y_2)\\ &= (\delta_{11}(y_1) - L_{x_1}(y_1)) + (\delta_{22}(y_2) - L_{x_2}(y_2)) + \delta_{12}(y_1) + \delta_{12}(y_2)\\ &\quad + \delta_{21}(y_1) + \delta_{21}(y_2) \in \text{Leib}(A).\end{aligned}$$

Thus, δ is an inner which completes the proof. □

Example 4.12. Consider the Leibniz algebra $\mathbf{A} = \mathbf{A}_1 \oplus \mathbf{A}_2$ where $\mathbf{A}_1 = \text{span}\{x, y, z\}$ and $\mathbf{A}_2 = \text{span}\{a, b, c\}$ with the non-zero multiplications in $\mathbf{A}$ given by $[x,z] = \alpha z$, $\alpha \in \mathbb{F} \setminus \{0\}$, $[x,y] = y$, $[y,x] = -y$, $[a,a] = c$, $[a,b] = b$ and $[b,a] = -b$. It is easy to see that $Z(\mathbf{A} / \text{Leib}(\mathbf{A}))$, $Z(\mathbf{A}_1 / \text{Leib}(\mathbf{A}_1))$ and $Z(\mathbf{A}_2 / \text{Leib}(\mathbf{A}_2))$ are trivial. From Example 3.19, we know that $\text{Der}(\mathbf{A}) = \text{span}\{\delta_1, \delta_2, \delta_3, \delta_4, \delta_5, \delta_6, \delta_7\}$ where

$\delta_1(x) = y$,	$\delta_1(y) = 0$,	$\delta_1(z) = 0$,	$\delta_1(a) = 0$,	$\delta_1(b) = 0$,	$\delta_1(c) = 0$,
$\delta_2(x) = 0$,	$\delta_2(y) = y$,	$\delta_2(z) = 0$,	$\delta_2(a) = 0$,	$\delta_2(b) = 0$,	$\delta_2(c) = 0$,
$\delta_3(x) = 0$,	$\delta_3(y) = 0$,	$\delta_3(z) = z$,	$\delta_3(a) = 0$,	$\delta_3(b) = 0$,	$\delta_3(c) = 0$,
$\delta_4(x) = 0$,	$\delta_4(y) = 0$,	$\delta_4(z) = 0$,	$\delta_4(a) = b$,	$\delta_4(b) = 0$,	$\delta_4(c) = 0$,
$\delta_5(x) = 0$,	$\delta_5(y) = 0$,	$\delta_5(z) = 0$,	$\delta_5(a) = c$,	$\delta_5(b) = 0$,	$\delta_5(c) = 0$,
$\delta_6(x) = 0$,	$\delta_6(y) = 0$,	$\delta_6(z) = 0$,	$\delta_6(a) = 0$,	$\delta_6(b) = b$,	$\delta_6(c) = 0$,
$\delta_7(x) = c$,	$\delta_7(y) = 0$,	$\delta_7(z) = 0$,	$\delta_7(a) = 0$,	$\delta_7(b) = 0$,	$\delta_7(c) = 0$.

Observe that $\text{im}(\delta_1 - L_{-y})$, $\text{im}(\delta_2 - L_x)$, $\text{im}(\delta_3 - L_0)$, $\text{im}(\delta_4 - L_{-b})$, $\text{im}(\delta_5 - L_a)$, $\text{im}(\delta_6 - L_a)$, $\text{im}(\delta_7 - L_0) \subseteq \text{Leib}(\mathbf{A})$. Therefore, $\text{Der}(\mathbf{A}) = \text{IDer}(\mathbf{A})$, i.e., $\mathbf{A}$ is complete. Moreover, we can see that $\text{Der}(\mathbf{A}_1) = \text{IDer}(\mathbf{A}_1)$ and $\text{Der}(\mathbf{A}_2) = \text{IDer}(\mathbf{A}_2)$. Hence $\mathbf{A}_1$ and $\mathbf{A}_2$ are also complete.

In (11), Rakhimov, Masutova, and Omirov established that every derivation of a simple Leibniz algebra can be expressed as a combination of three derivations. Here, we provide an alternative approach to this proof, specifically adapted for semisimple Leibniz algebras.

Theorem 4.13. (18) Let $\mathbf{A}$ be a semisimple Leibniz algebra. Then any derivation d of $\mathbf{A}$ can be written as $d = L_a + \alpha + \delta$ where $a \in S$, α: $\text{Leib}(\mathbf{A}) \to \text{Leib}(\mathbf{A})$, δ: $S \to \text{Leib}(\mathbf{A})$ where S is a semisimple Lie algebra and $\alpha([x,y]) = [x,\alpha(y)]$ for all $x, y \in \mathbf{A}$. Moreover, if $\mathbf{A}$ is simple, then the α is either zero or $\alpha(\text{Leib}(\mathbf{A})) = \text{Leib}(\mathbf{A})$.

Proof. Let A be a semisimple Leibniz algebra. By Theorem 2.18, A = S + Leib(A) where S is a semisimple Lie algebra. Since L(A) = $L(S)$ + L(Leib(A)) and L(Leib(A)) = {0}, then L(A) = $L(S)$. By [(9), Theorem 3.3], A is complete, and so Der(A) = IDer(A). Let $d \in$ Der(A). By Theorem 3.7, $d = L_a + h$ for some $a \in S$ and $h \in I$. Set $\alpha = h|_{\text{Leib(A)}}$ and $\delta = h|_S$. Then we can extend $\alpha$to be a derivation on A by defining $\alpha(x + y) = \alpha(y)$ for any $x \in S$ and $y \in$ Leib(A). Similarly, we can extend δ to be a derivation on A by defining $\delta(x + y) = \delta(x)$ for any $x \in S$ and $y \in$ Leib(A). Thus, $d = L_a + \alpha + \delta$, α(Leib(A)) ⊆ Leib(A) and $\delta(S)$ ⊆ Leib(A) as Leib(A) is a characteristic ideal of A. Since Leib(A) ⊆ Z^ℓ(A), $\alpha([x,y]) = [\alpha(x),y] - [x, \alpha(y)] = [x, \alpha(y)]$ for all $x, y \in$ A. If A is simple, then α(Leib(A)) is either {0} or Leib(A) which implies that α is either zero or α(Leib(A)) = Leib(A). □

Example 4.14. Let S = span{e, f, h} ⊕ span{a, b, c} and V = span{x, y}. Define A = $S \oplus V$ with brackets in A given by $[e,f] = h, [f,e] = -h, [h,e] = 2e, [e,h] = -2e, [h,f] = -2f, [f,h] = 2f, [e,y] = x, [f,x] = y, [h,x] = x, [h,y] = -y, [a,b] = c, [b,a] = -c, [c,a] = 2a, [a,c] = -2a, [c,b] = -2b, [b,c] = 2b$. Then A is a semisimple Leibniz algebra with Leib(A) = V . By direct calculations, we have that Der(A) = span{$d_1, d_2, d_3, d_4, d_5, d_6, d_7$} = IDer(A) where

$d_1(e) = e$, $d_1(f) = -f$, $d_1(h) = 0$, $d_1(x) = x$, $d_1(y) = 0$, $d_1(a) = 0$, $d_1(b) = 0$, $d_1(c) = 0$,

$d_2(e) = -e$, $d_2(f) = f$, $d_2(h) = 0$, $d_2(x) = 0$, $d_2(y) = y$, $d_2(a) = 0$, $d_2(b) = 0$, $d_2(c) = 0$,

$d_3(e) = 0$, $d_3(f) = h$, $d_3(h) = -2e$, $d_3(x) = 0$, $d_3(y) = x$, $d_3(a) = 0$, $d_3(b) = 0$, $d_3(c) = 0$,

$d_4(e) = -h$, $d_4(f) = 0$, $d_4(h) = 2f$, $d_4(x) = y$, $d_4(y) = 0$, $d_4(a) = 0$, $d_4(b) = 0$, $d_4(c) = 0$,

$d_5(e) = 0$, $d_5(f) = 0$, $d_5(h) = 0$, $d_5(x) = 0$, $d_5(y) = 0$, $d_5(a) = a$, $d_5(b) = -b$, $d_5(c) = 0$,

$d_6(e) = 0$, $d_6(f) = 0$, $d_6(h) = 0$, $d_6(x) = 0$, $d_6(y) = 0$, $d_6(a) = 0$, $d_6(b) = c$, $d_6(c) = -2a$,

$d_7(e) = 0$, $d_7(f) = 0$, $d_7(h) = 0$, $d_7(x) = 0$, $d_7(y) = 0$, $d_7(a) = c$, $d_7(b) = 0$, $d_7(c) = -2b$.

Since $d_1 - d_2 = L_h, d_3 = L_e, d_4 = L_f, d_5 = L_{-c/2}, d_6 = L_a, d_7 = L_b$, we have L(A) = span{$d_1 - d_2, d_3, d_4, d_5, d_6, d_7$} = $L(S)$. Let $k = d_1 + d_2$. Then $k \in I$ and $d_1 = L_{h/2} + k|_V + k|_S$ and $d_2 = L_{-h/2} + k|_V + k|_S$.

In (6), Meng proved that the following three statements are equivalent.

Proposition 4.15. (6) Let **L** be a Lie algebra. Then the following conditions are equivalent:

(i) **L** is complete.

(ii) Any extension G by **L** is a trivial extension, and $G = \mathbf{L} \oplus Z_G(\mathbf{L})$ where $Z_G(\mathbf{L}) = \{a \in G \mid [a,x] = 0 \text{ for all } x \in \mathbf{L}\}$.

(iii) hol(**L**) has the decomposition, $\mathrm{hol}(\mathbf{L}) = \mathbf{L} \oplus Z_{\mathrm{hol(L)}}(\mathbf{L})$ *where* $Z_{\mathrm{hol(L)}}(\mathbf{L}) = \{a \in \mathrm{hol}(\mathbf{L}) \mid [a,x] = 0 \text{ for all } x \in \mathbf{L}\}$.

Recently, in (9), Boyle, Misra and Stitzinger proved some analog for Leibniz algebras.

Theorem 4.16. (9) Let **A** be a Leibniz algebra. Then **A** is complete if and only if $\mathrm{hol}(\mathbf{A}) = \mathbf{A} + (Z^{\ell}_{\mathrm{hol(A)}}(\mathbf{A}) \oplus I)$ and $\mathbf{A} \cap (Z^{\ell}_{\mathrm{hol(A)}}(\mathbf{A}) \oplus I) = \mathrm{Leib}(\mathbf{A})$ where $I = \{\delta \in \mathrm{Der}(\mathbf{A}) \mid \mathrm{im}(\delta) \subseteq \mathrm{Leib}(\mathbf{A})\}$.

In the following theorem, we prove another necessary and sufficient condition for the Leibniz algebra **A** to be complete.

Theorem 4.17. Let **A** be a Leibniz algebra. Then the following conditions are equivalent:

(i) **A** is complete.

(ii) For any extension B of **A**, $B = \mathbf{A} + X$ where $X = \{x \in B \mid [x, \mathbf{A}] \subseteq \mathrm{Leib}(\mathbf{A})\}$ and $\mathbf{A} \cap X = \mathrm{Leib}(\mathbf{A})$.

Proof. (i) $\Rightarrow$ (ii) Suppose that **A** is complete. Let B be an extension of **A**. It is clear that $\mathrm{Leib}(\mathbf{A}) \subseteq \mathbf{A} \cap X$. To show that $\mathbf{A} \cap X \subseteq \mathrm{Leib}(\mathbf{A})$, we let $x \in \mathbf{A} \cap X$. Then for all $a \in \mathbf{A}$, $[x + \mathrm{Leib}(\mathbf{A}), a + \mathrm{Leib}(\mathbf{A})] = [x,a] + \mathrm{Leib}(A) = \mathrm{Leib}(\mathbf{A})$ and hence $x + \mathrm{Leib}(\mathbf{A}) \in Z(\mathbf{A} / \mathrm{Leib}(\mathbf{A})) = \{0\}$ which implies $x \in \mathrm{Leib}(\mathbf{A})$. Therefore, $\mathbf{A} \cap X = \mathrm{Leib}(\mathbf{A})$. Let $x \in B$. Since **A** is an ideal of B, $\mathrm{ad}_x|_{\mathbf{A}} \in \mathrm{Der}(\mathbf{A})$. Thus, there exists $b \in \mathbf{A}$ such that $\mathrm{im}(\mathrm{ad}_x|_{\mathbf{A}} - L_b) \subseteq \mathrm{Leib}(\mathbf{A})$. So, we have

$[x - b, \mathsf{A}] \subseteq \mathrm{Leib}(\mathsf{A})$ and $x - b \in X$. Hence, $x \in \mathsf{A} + X$ which implies $B \subseteq \mathsf{A} + X$. Since the reverse inclusion is clear, we have $B = \mathsf{A} + X$.

(ii) ⇒ (i) Suppose that (ii) holds. Since hol(A) is an extension of A, hol(A) = A + X where $X = \{x + \delta \in \mathrm{hol}(\mathsf{A}) \mid [x + \delta, \mathsf{A}] = [x, \mathsf{A}] + \delta(\mathsf{A}) \subseteq \mathrm{Leib}(\mathsf{A})\}$. Set $I = \{ \delta \in \mathrm{Der}(\mathsf{A}) \mid \mathrm{im}(\delta) \subseteq \mathrm{Leib}(\mathsf{A})\}$. To show $\mathsf{A} + (Z^{\ell}_{\mathrm{hol}(\mathsf{A})}(\mathsf{A}) \oplus I) \subseteq \mathsf{A} + X$, let $a \in \mathsf{A} + (Z^{\ell}_{\mathrm{hol}(\mathsf{A})}(\mathsf{A}) \oplus I)$. Then by Proposition 3.12, $a = b + c - L_c + \delta$ for some $b, c \in \mathsf{A}$ and $\delta \in I$. For all $d \in \mathsf{A}$, we have $[c - L_c + \delta, d] = [c,d] - L_c(d) + \delta(d) = \delta(d) \in \mathrm{Leib}(\mathsf{A})$. Hence $c - L_c + \delta \in X$ which implies $a = b + c - L_c + \delta \in \mathsf{A} + X$. Conversely, let $a \in \mathsf{A} + X = \mathrm{hol}(\mathsf{A}) = \mathsf{A} \oplus \mathrm{Der}(\mathsf{A})$. Then $a = b + c + \delta_1 = d + \delta_2$ for some $b, d \in \mathsf{A}$, $c + \delta_1 \in X$ and $\delta_2 \in \mathrm{Der}(\mathsf{A})$. This implies $\delta_1 = \delta_2$ and $b + c = d$. Note that $\mathrm{im}(L_c + \delta_2) = \mathrm{im}(L_c + \delta_1) \subseteq \mathrm{Leib}(\mathsf{A})$ and hence $L_c + \delta_2 \in I$. Thus, $a = d + \delta_2 = d - c + c - L_c + L_c + \delta_2 \in \mathsf{A} + (Z^{\ell}_{\mathrm{hol}(\mathsf{A})}(\mathsf{A}) \oplus I)$ which implies $\mathsf{A} + X \subseteq \mathsf{A} + (Z^{\ell}_{\mathrm{hol}(\mathsf{A})}(\mathsf{A}) \oplus I)$. Therefore, we have that $\mathrm{hol}(\mathsf{A}) = \mathsf{A} + X \subseteq \mathsf{A} + (Z^{\ell}_{\mathrm{hol}(\mathsf{A})}(\mathsf{A}) \oplus I)$. Also, we have that $\mathsf{A} \cap (Z^{\ell}_{\mathrm{hol}(\mathsf{A})}(\mathsf{A}) \oplus I) = \mathsf{A} \cap X = \mathrm{Leib}(\mathsf{A})$. By Theorem 4.16, it follows that A is complete. □

Therefore, by Theorem 4.16 and Theorem 4.17, we have the following full Leibniz algebra analog of the Lie algebra result given in (6).

Corollary 4.18. Let A be a Leibniz algebra. Then the following conditions are equivalent:

(i) A is complete.

(ii) For any extension B of A, $B = \mathsf{A} + X$ where $X = \{x \in B \mid [x, \mathsf{A}] \subseteq \mathrm{Leib}(\mathsf{A})\}$ and $\mathsf{A} \cap X = \mathrm{Leib}(\mathsf{A})$.

(iii) $\mathrm{hol}(\mathsf{A}) = \mathsf{A} + (Z^{\ell}_{\mathrm{hol}(\mathsf{A})}(\mathsf{A}) \oplus I)$ and $\mathsf{A} \cap (Z^{\ell}_{\mathrm{hol}(\mathsf{A})}(\mathsf{A}) \oplus I) = \mathrm{Leib}(\mathsf{A})$ where $I = \{ \delta \in \mathrm{Der}(\mathsf{A}) \mid \mathrm{im}(\delta) \subseteq \mathrm{Leib}(\mathsf{A})\}$.

In (21), Ayupov, Khudoyberdiyev and Shermatova gave a conjecture that if a complete Leibniz algebra A is an ideal of the Leibniz algebra B, then $B = \mathsf{A} \oplus I$, where I is an ideal of B. By our definition of completeness, this conjecture is not true as shown in the following example.

Example 4.19. Consider the Leibniz algebra B = span$\{x, y, z, a\}$ with non-zero brackets defined by $[x,y] = y$, $[y,x] = -y$, $[x,z] = 2z$ and $[x,a] = z$ and the complete Leibniz algebra $\mathbf{A}$ = span$\{x, y, z\}$ with non-zero brackets defined by $[x,y] = y, [y,x] = -y, [x,z] = 2z$. Then $\mathbf{A}$ is an ideal of B and $B = \mathbf{A} \oplus$ span$\{a\}$. However, span$\{a\}$ is not an ideal of B.

In (6), Meng proved that if a Lie algebra $\mathbf{L}$ has a trivial center and ad($\mathbf{L}$) is a characteristic ideal of Der($\mathbf{L}$), then Der($\mathbf{L}$) is a complete Lie algebra. Consequently, for a complete Lie algebra $\mathbf{L}$, Der($\mathbf{L}$) is also complete. However, this statement does not hold for some Leibniz algebras. The following example illustrates that there exists a complete Leibniz algebra $\mathbf{A}$ for which Der($\mathbf{A}$) is not complete.

Example 4.20. (12) From Example 4.3, for the complete Leibniz algebra $\mathbf{A}$ = span$\{x, y, z\}$ with non-zero brackets defined by $[x,z] = \alpha z$, $[x,y] = y$ and $[y,x] = -y$ for some $\alpha \in \mathbb{F} \setminus \{0\}$, we have that Der($\mathbf{A}$) = span$\{\delta_1, \delta_2, \delta_3\}$ where

$$\delta_1(x) = y, \quad \delta_1(y) = 0, \quad \delta_1(z) = 0,$$

$$\delta_2(x) = 0, \quad \delta_2(y) = y, \quad \delta_2(z) = 0,$$

$$\delta_3(x) = 0, \quad \delta_3(y) = 0, \quad \delta_3(z) = z.$$

Since $[\delta_1, \delta_2] = -\delta_1$, $[\delta_1, \delta_3] = 0$ and $[\delta_2, \delta_3] = 0$, we have that $\delta_3 \in Z(\mathrm{Der}(\mathbf{A}))$ which implies that $Z(\mathrm{Der}(\mathbf{A})) \neq \{0\}$. Therefore, Der($\mathbf{A}$) is not complete.

Recall that $I = \{\delta \in \mathrm{Der}(\mathbf{A}) \mid \mathrm{im}(\delta) \subseteq \mathrm{Leib}(\mathbf{A})\}$ is an ideal of Der($\mathbf{A}$). The following theorem is one of our main results.

Theorem 4.21. (18) Let $\mathbf{A}$ be a complete Leibniz algebra. If $\mathbf{A}$ / Leib($\mathbf{A}$) is a complete Lie algebra, then Der($\mathbf{A}$) / I is complete and Der($\mathbf{A}$) / $I \cong$ Der($\mathbf{A}$ / Leib($\mathbf{A}$)).

Proof. Let $\mathbf{A}$ be a complete Leibniz algebra. Suppose that $\mathbf{A}$ / Leib($\mathbf{A}$) is a complete Lie algebra. Then Der($\mathbf{A}$ / Leib($\mathbf{A}$)) is complete. Define a linear map φ: Der($\mathbf{A}$) $\rightarrow$ Der($\mathbf{A}$ / Leib($\mathbf{A}$)) by $\varphi(\delta) = \delta'$ where $\delta'(x+\mathrm{Leib}(\mathbf{A})) = \delta(x) + \mathrm{Leib}(\mathbf{A})$ for any $\delta \in \mathrm{Der}(\mathbf{A})$ and $x \in \mathbf{A}$. Let $\delta_1, \delta_2 \in \mathrm{Der}(\mathbf{A})$. Then for all $x \in \mathbf{A}$,

$$\begin{aligned}\varphi([\delta_1, \delta_2])(x + \text{Leib}(A)) &= \varphi(\delta_1\delta_2 - \delta_2\delta_1)(x + \text{Leib}(A)) \\ &= (\delta_1\delta_2 - \delta_2\delta_1)'(x + \text{Leib}(A)) \\ &= (\delta_1\delta_2)(x) - (\delta_2\delta_1)(x) + \text{Leib}(A) \\ &= \delta_1(\delta_2(x)) - \delta_2(\delta_1(x)) + \text{Leib}(A) \\ &= \delta'_1(\delta_2(x) + \text{Leib}(A)) - \delta'_2(\delta_1(x) + \text{Leib}(A)) \\ &= \delta'_1(\delta'_2(x + \text{Leib}(A)) - \delta'_2(\delta'_1(x + \text{Leib}(A)) \\ &= (\delta'_1\delta'_2 - \delta'_2\delta'_1)(x + \text{Leib}(A)) \\ &= [\varphi(\delta_1), \varphi(\delta_2)](x + \text{Leib}(A)).\end{aligned}$$

Hence, $\varphi([\delta_1, \delta_2]) = [\varphi(\delta_1), \varphi(\delta_2)]$. Clearly, $I = \{d \in \text{Der}(A) \mid \text{im}(d) \subseteq \text{Leib}(A)\} \subseteq \ker(\varphi)$. Let $\delta \in \ker(\varphi)$. Then for all $x \in A$, $\delta(x) + \text{Leib}(A) = \delta'(x + \text{Leib}(A)) = \text{Leib}(A)$ hence $\delta(x) \in \text{Leib}(A)$ which implies that $\delta \in I$. Therefore, $\ker(\varphi) = I$. To show that φ is onto, let $\delta' \in \text{Der}(A / \text{Leib}(A))$. Since $A / \text{Leib}(A)$ is a complete Lie algebra, there exists $a + \text{Leib}(A) \in A / \text{Leib}(A)$ such that $\delta' = L_{a + \text{Leib}(A)}$. Then $L_a(x) + \text{Leib}(A) = [a,x] + \text{Leib}(A) = [a + \text{Leib}(A), x + \text{Leib}(A)] = L_{a + \text{Leib}(A)}(x + \text{Leib}(A))$. This implies that $\varphi(L_a) = \delta'$ and φ is onto. Hence $\text{Der}(A) / I \cong \text{Der}(A / \text{Leib}(A))$. □

In (9), it is proved that for a Leibniz algebra A, if A / Leib(A) is a complete Lie algebra, then A is complete. We examine the Leibniz algebra A such that A / Leib(A) is complete and obtain the following results.

Corollary 4.22. (18) Let A be a Leibniz algebra such that A / Leib(A) is a complete Lie algebra. Then

(i) $I_A = \text{Leib}(A)$ and A / I_A is a complete Lie algebra,

(ii) $\text{hol}(A)/(I_A \oplus I)$ is a complete Lie algebra,

(iii) $(L(A) + I)/I \cong \text{IDer}(A)/I \cong \text{Der}(A)/I \cong \text{Der}(A/\text{Leib}(A)) \cong A/\text{Leib}(A)$,

(iv) $\dim(\text{Leib}(A)) = \dim(I_A) = \dim(Z^{\ell}(A)) = \dim(A) - \dim(L(A)) = \dim(A) + \dim(I) - \dim(\text{Der}(A))$.

Proof. Let A be a Leibniz algebra. Assume that A / Leib(A) is a complete Lie algebra.

(i) Since A is complete, by Corollary 3.5, we have that A / $I_A \cong L$(A / Leib(A)) = ad(A / Leib(A)) $\cong$ A / Leib(A). Hence, I_A = Leib(A) and A / I_A is a complete Lie algebra.

(ii) By Proposition 3.11, we have hol(A) / ($I_A \oplus I$) = A / $I_A \oplus$ Der(A) / I. By (i), A / I_A is complete and by Theorem 4.21, Der(A) / I is complete. Therefore, by Theorem 4.11, hol(A) / ($I_A \oplus I$) is complete.

(iii) Since A is complete by (9), it follows that Der(A) = IDer(A) = L(A) + I. Then the statement holds.

(iv) The results follow immediately from (i), (iii) and Theorem 3.4. □

Example 4.23. Consider the Leibniz algebra A = span{x, y, z} with non-zero multiplications defined by $[x,y] = y$, $[y,x] = -y$ and $[x,x] = z$. From Example 3.9, we have that Der(A) = span{d_1, d_2, d_3} = IDer(A) where

$$d_1(x) = y, \quad d_1(y) = 0, \quad d_1(z) = 0,$$
$$d_2(x) = z, \quad d_2(y) = 0, \quad d_2(z) = 0,$$
$$d_3(x) = 0, \quad d_3(y) = y, \quad d_3(z) = 0.$$

Since Z(A / Leib(A)) is trivial, A is complete. By (12), it is known that A / Leib(A) and Der(A) / I are complete Lie algebras. In this case, we have I_A = span{z} = Leib(A) and I = span{d_2}. Thus, dim(Der(A)) = 3 = 3 − 1 + 1 = dim(A) − dim(Leib(A)) + dim(I).

CHAPTER 5
GENERALIZATIONS OF DERIVATIONS OF LEIBNIZ ALGEBRAS

In 2021, Chang, Chen and Zhang (21) studied a generalization of derivations of finite dimensional Lie algebras over an algebraically closed field of characteristic zero. Specifically, they introduced the notion of (σ,τ)-derivations which connects with the automorphism group when specializing in the case where σ and τ are automorphisms. Motivated by these results, we introduce the notion of a generalization of derivations of Leibniz algebras and explore its properties. Let **A** be a Leibniz algebra. We denote Aut(**A**) to be the automorphism group of **A**.

Definition 5.1. Let G be a subgroup of Aut(**A**). A linear map D : **A** $\longrightarrow$ **A** is called a *G-derivation* of **A** if there exist two automorphisms σ, $\tau \in G$ such that $D[x,y] = [D(x), \sigma(y)] + [\tau(x), D(y)]$ for all $x,y \in$ **A**. In this case, σ and τ are called *associated* automorphisms of D.

We denote Der_G(**A**) to be the set of all G-derivations of **A**. Given two elements σ, $\tau \in G$, we denote $\mathrm{Der}_{\sigma,\tau}$(**A**) to be the set of all G-derivations associated to σ and τ. Clearly, $\mathrm{Der}_{\sigma,\tau}$(**A**) $\subseteq$ Der_G(**A**) is a vector space and in particular, $\mathrm{Der}_{\mathrm{id},\mathrm{id}}$(**A**) = Der(**A**). For simplicity of notation, we denote $\mathrm{Der}_{\sigma,\mathrm{id}}$(**A**) as Der_{σ}(**A**). The following proposition is the Leibniz algebra analogue of the result in [21, Proposition 2.1].

Proposition 5.2. Let **A** be a Leibniz algebra and let σ, $\tau \in$ Aut(**A**). Then $\mathrm{Der}_{\sigma,\tau}$**A**) $\cong$ $\mathrm{Der}_{\tau^{-1}\sigma}$(**A**).

Proof. Define a map $\varphi_\tau : \mathrm{Der}_{\sigma,\tau}(\mathbf{A}) \longrightarrow \mathrm{Der}_{\tau^{-1}\sigma}(\mathbf{A})$ by $\varphi_\tau(D) = \tau^{-1} \circ D$ for all $D \in \mathrm{Der}_{\sigma,\tau}(\mathbf{A})$. Since $D \in \mathrm{Der}_{\sigma,\tau}(\mathbf{A})$, for any x, $y \in$ **A**, $D[x,y] = [D(x), \sigma(y)] + [\tau(x), D(y)]$. It follows that $\tau^{-1} \circ D([x,y]) = \tau^{-1}(D[x,y]) = \tau^{-1}([D(x), \sigma(y)] + [\tau(x), D(y)]) = [\tau^{-1} \circ D(x), \tau^{-1} \circ \sigma(y)] + [\tau^{-1} \circ \tau(x), \tau^{-1} \circ D(y)]$ and hence $\tau^{-1} \circ D \in \mathrm{Der}_{\tau^{-1}\sigma}(\mathbf{A})$. For all D_1, $D_2 \in \mathrm{Der}_{\sigma,\tau}(\mathbf{A})$ and $\alpha \in \mathbb{F}$, we have that $\tau^{-1} \circ (\alpha D_1 + D_2) = \alpha(\tau^{-1} \circ D_1) + \tau^{-1} \circ D_2$. Hence φ_τ is a linear map. Define a map $\phi_\tau : \mathrm{Der}_{\tau^{-1}\sigma}(\mathbf{A})$

$\rightarrow \mathrm{Der}_{\sigma,\tau}$(A) by $\phi_\tau(D) = \tau \circ D$ for all $D \in \mathrm{Der}_{\tau^{-1}\sigma}$(A). Then ϕ_τ is also linear and $\varphi_{\tau^{-1}} = \phi_\tau$. Therefore, φ_τ is an isomorphism and $\mathrm{Der}_{\sigma,\tau}(\mathrm{A}) \cong \mathrm{Der}_{\tau^{-1}\sigma}(\mathrm{A})$. □

By Proposition 5.2, the study of $\mathrm{Der}_{\sigma,\tau}$(A) with two automorphisms σ, τ can be turned to the study of $\mathrm{Der}_{\sigma'}$(A) with one automorphism σ'. In the case that $\sigma = \tau$, we have that $\mathrm{Der}_{\sigma,\sigma}(\mathrm{A}) \cong \mathrm{Der}_{\sigma^{-1}\sigma} = \mathrm{Der}(\mathrm{A})$. In general, the vector space Der_σ(A) may not be a Lie subalgebra of gl(A). The following proposition shows that under some conditions, Der_σ(A) and Der(A) coincide. It is the Leibniz algebra analogue of the result in [(21), Proposition 2.4].

Proposition 5.3. Let A be a Leibniz algebra and let $\sigma, \tau \in \mathrm{Aut}(\mathrm{A})$. If $\mathrm{im}(\sigma - \tau) \subseteq Z(\mathrm{A})$, then $\mathrm{Der}_\sigma(\mathrm{A}) = \mathrm{Der}_\tau(\mathrm{A})$. In particular, if $\mathrm{im}(\sigma - \mathrm{id}) \subseteq Z(\mathrm{A})$, then $\mathrm{Der}_\sigma(\mathrm{A}) = \mathrm{Der}(\mathrm{A})$.

Proof. Assume that $\mathrm{im}(\sigma - \tau) \subseteq Z(\mathrm{A})$. Then for all $a \in \mathrm{A}$, $\sigma(a) - \tau(a) \in Z(\mathrm{A})$. For any $x \in \mathrm{A}$, $D \in \mathrm{Der}(\mathrm{A})$, we have that $[D(x), \sigma(a) - \tau(a)] = 0$ which implies that $[D(x), \sigma(a)] = [D(x), \tau(a)]$. Therefore, for any $D \in \mathrm{Der}_\sigma(\mathrm{A})$, we have that $D[x,y] = [D(x), \sigma(y)] + [x,D(y)] = [D(x), \tau(y)] + [x,D(y)]$ which implies that $D \in \mathrm{Der}_\tau(\mathrm{A})$. Thus, $\mathrm{Der}_\sigma(\mathrm{A}) \subseteq \mathrm{Der}_\tau(\mathrm{A})$. It can be shown similarly that $\mathrm{Der}_\tau(\mathrm{A}) \subseteq \mathrm{Der}_\sigma(\mathrm{A})$. Therefore, $\mathrm{Der}_\sigma(\mathrm{A}) = \mathrm{Der}_\tau(\mathrm{A})$. Clearly, if $\tau = \mathrm{id}$, then $\mathrm{Der}_\sigma(\mathrm{A}) = \mathrm{Der}(\mathrm{A})$. □

The following results are the Leibniz algebra analogue of the results in [(21), Lemma 3.21 and Proposition 2.6].

Lemma 5.4. Let A be a Leibniz algebra, $\sigma \in \mathrm{Aut}(\mathrm{A})$ and $D \in \mathrm{Der}_\sigma(\mathrm{A})$. Then $[D,L_x] = \sigma \circ L_{\sigma^{-1}\circ D(x)}$ for all $x \in \mathrm{A}$.

Proof. Let $x, y \in \mathrm{A}$. Then $[D,L_x](y) = D \circ L_x(y) - L_x \circ D(y) = D[x,y] - [x,D(y)] = [D(x), \sigma(y)] + [x,D(y)] - [x,D(y)] = [D(x), \sigma(y)] = \sigma[\sigma^{-1}(D(x)),y]) = \sigma \circ L_{\sigma^{-1}\circ D(x)}(y)$ Thus, $[D,L_x] = \sigma\circ L_{\sigma^{-1}\circ D(x)}$.

□

Proposition 5.5. *Let* A *be a Leibniz algebra such that* $A^2 \neq 0$*. If* $\sigma \in \mathrm{Aut}(A)$ *and* $D \in \mathrm{Der}_\sigma(A)$ *such that* $[D, \sigma](A) \subseteq Z(A)$*, then* $A^2 \subseteq \ker([D, \sigma])$.

Proof. Assume that $A^2 \neq \{0\}$. Let $\sigma \in \mathrm{Aut}(A)$ and $D \in \mathrm{Der}_\sigma(A)$ such that $[D, \sigma](A) \subseteq Z(A)$. Then for any $x, y \in A$, we have that

$$\begin{aligned}
[D, \sigma]([x,y]) &= D(\sigma([x,y])) - \sigma(D([x,y])) \\
&= D([\sigma(x), \sigma(y)]) - \sigma([D(x), \sigma(y)] + [x,D(y)]) \\
&= [D(\sigma(x)), \sigma(\sigma(y))] - [\sigma(x),D(\sigma(y))] - [\sigma(D(x)), \sigma(\sigma(y))] + [\sigma(x), \sigma(D(y))] \\
&= [D(\sigma(x)) - \sigma(D(x)), \sigma^2(y)] + [\sigma(x),D(\sigma(y)) - \sigma(D(y))] \\
&= [[D, \sigma](x), \sigma^2(y)] + [\sigma(x),[D, \sigma](y)] \\
&= 0.
\end{aligned}$$

Hence, $A^2 \subseteq \ker([D, \sigma])$. □

In 2017, Said Husain, Rakhimov and Basri (22) studied centroids of Leibniz algebras and their properties. Here, we investigate comparisons and connections between *G*-derivations and centroids.

Definition 5.6. (22) Let A be a Leibniz algebra. The *centroid* C(A) of A is the set of all linear maps $D : A \rightarrow A$ such that $D[x,y] = [D(x),y] = [x,D(y)]$ for all $x, y \in A$.

We obtain the following result which is the Leibniz algebra analogue of the result in [(21), Proposition 3.15].

Proposition 5.7. Let A be a Leibniz algebra, $\sigma \in \mathrm{Aut}(A)$ and $D \in C(A) \cap \mathrm{Der}_\sigma(A)$. Then $L_{D(x)} = 0$ for all $x \in A$. In particular, if $Z^\ell(A) = 0$, then $C(A) \cap \mathrm{Der}(A) = \{0\}$.

Proof. Let A be a Leibniz algebra, $\sigma \in \mathrm{Aut}(A)$ and $D \in C(A) \cap \mathrm{Der}_\sigma(A)$. Then for any $x, y \in A$, we have that $D[x,y] = [D(x), \sigma(y)] + [x,D(y)]$. Since $D \in C(A)$, $[x,D(y)] = [D(x), \sigma(y)] + [x,D(y)]$ and so $[D(x), \sigma(y)] = 0$. Hence, $[D(x), A] = 0$ because $\sigma \in \mathrm{Aut}(A)$. This means that $L_{D(x)} = 0$ for all $x \in A$. In particular, if $Z^\ell(A) = 0$, then $D(x) = 0$ for all $x \in A$ hence $D = 0$. Therefore, $C(A) \cap \mathrm{Der}(A) = \{0\}$. □

We consider a subalgebra M of A and an automorphism of A such that $\sigma(M) \subseteq M$. We denote $\mathrm{Der}_{\sigma,M}(\mathrm{A})$ to be the set of all σ-derivations of A which stabilizes M, i.e., $\mathrm{Der}_{\sigma,M}(\mathrm{A}) = \{D \in \mathrm{Der}_{\sigma}(\mathrm{A}) \mid D(M) \subseteq M\}$. The following proposition is the Leibniz algebra analogue of the result in [(21), Proposition 3.17].

Proposition 5.8. *Let* A *be a Leibniz algebra. Let* M *be a subalgebra of* A *and* $\sigma \in \mathrm{Aut}(\mathrm{A})$ *such that* $\sigma(M) \subseteq M$. *Then* $\mathrm{Der}_{\sigma,M}(\mathrm{A})$ *is a subspace of* $\mathrm{Der}_{\sigma}(\mathrm{A})$. *Moreover, if* M *is an ideal of* A *and* $M^2 = M$, *then* $\mathrm{Der}_{\sigma,M}(\mathrm{A}) = \mathrm{Der}_{\sigma}(\mathrm{A})$.

Proof. Let A be a Leibniz algebra. Let M be a subalgebra of A and $\sigma \in \mathrm{Aut}(\mathrm{A})$ such that $\sigma(M) \subseteq M$. Let $S, T \in \mathrm{Der}_{\sigma,M}(\mathrm{A})$ and $\alpha, \beta \in \mathbb{F}$. Clearly, $\alpha S + \beta T \in \mathrm{Der}_{\sigma}(\mathrm{A})$. Also, for any $a \in M$, we have that $(\alpha S + \beta T)(a) = \alpha S(a) + \beta T(a) \in M$. Thus, $\mathrm{Der}_{\sigma,M}(\mathrm{A})$ is a subspace of $\mathrm{Der}_{\sigma}(\mathrm{A})$. Assume that M is an ideal of A such that $M^2 = M$. To show that $\mathrm{Der}_{\sigma}(\mathrm{A}) \subseteq \mathrm{Der}_{\sigma,M}(\mathrm{A})$, let $D \in \mathrm{Der}_{\sigma}(\mathrm{A})$ and $a \in M$. Since $M = M^2$, there exist $b, c \in M$ such that $a = [b,c]$. Then $D(a) = D([b,c]) = [D(b), \sigma(c)] + [b,D(c)]$. Since $\sigma(c) \in M$ and M is an ideal of A, we have $D(a) \in M$ hence $D \in \mathrm{Der}_{\sigma,M}(\mathrm{A})$. This means that $\mathrm{Der}_{\sigma}(\mathrm{A}) \subseteq \mathrm{Der}_{\sigma,M}(\mathrm{A})$. Since the reverse inclusion is clear, $\mathrm{Der}_{\sigma,M}(\mathrm{A}) = \mathrm{Der}_{\sigma}(\mathrm{A})$. □

REFERENCES

1. Cheng T-P, Li L-F, Gross D. Gauge Theory of Elementary Particle Physics. Physics Today - PHYS TODAY. 1985;38.

2. Georgi H. Lie Algebras In Particle Physics: from Isospin To Unified Theories. Revised and updated edition. ed. United States: CRC Press; 1999.

3. Lipkin HJ, editor Lie groups for pedestrians1965.

4. Dobrev V. Lie Theory and Its Applications in Physics Varna, Bulgaria, June 2019: Varna, Bulgaria, June 20192020.

5. Loday J-L, editor Une version non commutative des algèbres de Lie : les algèbres de Leibniz1993.

6. Meng D. Some results on complete lie algebras. Communications in Algebra. 1994;22:5457-507.

7. Jacobson N, editor A note on automorphisms and derivations of Lie algebras1955.

8. Ancochea Bermúdez JM, Campoamor-Stursberg R. On a complete rigid Leibniz non-Lie algebra in arbitrary dimension. Linear Algebra and its Applications. 2013;438(8):3397-407.

9. Boyle K, Misra KC, Stitzinger E. Complete Leibniz algebras. Journal of Algebra. 2020;557:172-80.

10. Tôgô S. On the Derivation Algebras of Lie Algebras. Canadian Journal of Mathematics. 1961;13:201-16.

11. Rakhimov IS, Masutova KK, Omirov BA. On Derivations of Semisimple Leibniz Algebras. Bulletin of the Malaysian Mathematical Sciences Society. 2014;40:295-306.

12. Kongsomprach Y, Pongprasert S, Rungratgasame T, Tiansa-ard S. Completeness of low-dimensional Leibniz algebras: Annual Meeting in Mathematics 2023. Thai Journal of Mathematics. 2024;22(1):165–78-–78.

13. Jacobson N. Lie Algebras: Dover; 1979.

14. Ayupov S, Omirov B, Rakhimov I. Leibniz algebras: structure and classification: Chapman and Hall/CRC; 2019.

15. Barnes DW. Some Theorems on Leibniz Algebras. Communications in Algebra. 2011;39(7):2463-72.

16. Demir I, Misra K, Stitzinger E. On some structures of Leibniz algebras. Contemporary Mathematics. 2014;623:41-54.

17. Barnes D. On Levi's Theorem for Leibniz algebras. Bulletin of the Australian Mathematical Society. 2011;86.

18. Patlertsin S, Pongprasert S, Rungratgasame T. On Inner Derivations of Leibniz Algebras. Mathematics. 2024;12(8):1152.

19. Shermatova Z, Khudoyberdiyev A. On special subalgebras of derivations of Leibniz algebras. Algebra and Discrete Mathematics. 2022;34:326-36.

20. Biyogmam G, Tcheka C. A note on outer derivations of Leibniz algebras. Communications in Algebra. 2021;49:1-12.

21. Ayupov S, Khudoyberdiyev A, Shermatova Z. On complete Leibniz algebras. International Journal of Algebra and Computation. 2022;32:1-24.

Appendix A

For readers' convenience, Appendix A provides a list of notations and definitions used in this work.

A *Lie algebra* **L** is a vector space over $\mathbb{F}$ with a bilinear map $[\ ,\ ] : \mathbf{L} \times \mathbf{L} \longrightarrow \mathbf{L}$ such that following axioms are satisfied:

(i) $[a,a] = 0$ for all $a \in \mathbf{L}$ and

(ii) $[a,[b,c]] + [b,[c,a]] + [c,[a,b]] = 0$ for all $a, b, c \in \mathbf{L}$ (Jacobi Identity).

For a Lie algebra **L**, a derivation $d : \mathbf{L} \rightarrow \mathbf{L}$ is *inner* if there exists $x \in \mathbf{L}$ such that $d = \mathrm{ad}_x$ where $\mathrm{ad}_x : \mathbf{L} \longrightarrow \mathbf{L}$ is defined by $\mathrm{ad}_x(y) = [x,y]$ for all $y \in \mathbf{L}$. Otherwise, the derivation is called *outer*. A Lie algebra **L** is said to be *complete* if its center is trivial and all derivations are inner.

A *(left) Leibniz algebra* **A** is a vector space over $\mathbb{F}$ with a bilinear map (called bracket) $[\ ,\ ] : \mathbf{A} \times \mathbf{A} \rightarrow \mathbf{A}$ that satisfies the Leibniz identity

$$[a,[b,c]] = [[a,b],c] + [b,[a,c]] \text{ for all } a, b, c \in \mathbf{A}.$$

Let **A** be a Leibniz algebra. For subsets M and N of **A**, we define the *product* of M and N to be the subspace spanned by all brackets $[a,b]$, where $a \in M$ and $b \in N$, denoted by $[M,N]$. A subspace M of **A** is called a *subalgebra* of **A** if $[M,M] \subseteq M$. A subspace M of **A** is called an *ideal* of **A** if $[M,\mathbf{A}] \subseteq M$ and $[\mathbf{A},M] \subseteq M$. The *left center* of **A** is $Z^{\ell}(\mathbf{A}) = \{x \in \mathbf{A} \mid [x,a] = 0 \text{ for all } a \in \mathbf{A}\}$. The *right center* of **A** is $Z^{r}(\mathbf{A}) = \{x \in \mathbf{A} \mid [a,x] = 0 \text{ for all } a \in \mathbf{A}\}$. The *center* of **A** is $Z(\mathbf{A}) = Z^{\ell}(\mathbf{A}) \cap Z^{r}(\mathbf{A})$. We denote $\mathrm{Leib}(\mathbf{A}) = \mathrm{span}\{[x,x] \mid x \in \mathbf{A}\}$. For any ideal M of **A**, we define the *quotient space* by $\mathbf{A}/M = \{a + M \mid a \in \mathbf{A}\}$ with the bracket $[x + M, y + M] = [x,y] + M$, for all $x, y \in \mathbf{A}$.

A linear transformation $\delta: \mathbf{A} \rightarrow \mathbf{A}$ is a *derivation* of **A** if $\delta[a,b] = [\delta(a),b] + [a, \delta(b)]$ for all $a, b \in \mathbf{A}$. We denote $\mathrm{Der}(\mathbf{A})$ to be the set of all derivations of **A** with the commutator bracket $[\delta_1, \delta_2] := \delta_1 \circ \delta_2 - \delta_2 \circ \delta_1$ for any $\delta_1, \delta_2 \in \mathrm{Der}(\mathbf{A})$. An ideal M of $\mathbf{A}$ is a *characteristic ideal* if $\delta(M) \subseteq M$ for all $\delta \in \mathrm{Der}(\mathbf{A})$. For any $a \in \mathbf{A}$, we define the *left multiplication* operator $L_a : \mathbf{A} \to \mathbf{A}$ by $L_a(b) = [a,b]$ for all $b \in \mathbf{A}$. We denote $L(\mathbf{A}) = \mathrm{span}\{L_a \mid a \in \mathbf{A}\}$, $I = \{d \in \mathrm{Der}(\mathbf{A}) \mid \mathrm{im}(d) \subseteq \mathrm{Leib}(\mathbf{A})\}$, and $I_{\mathbf{A}} = \{x \in \mathbf{A} \mid \mathrm{im}(L_x) \subseteq \mathrm{Leib}(\mathbf{A})\}$. A derivation $\delta \in \mathrm{Der}(\mathbf{A})$ is said to be *inner* if there exists $x \in \mathbf{A}$ such that $\mathrm{im}(\delta - L_x) \subseteq \mathrm{Leib}(\mathbf{A})$. We denote $\mathrm{IDer}(\mathbf{A})$ be the set of all inner derivations of $\mathbf{A}$. A Leibniz algebra $\mathbf{A}$ is *complete* if $Z(\mathbf{A} / \mathrm{Leib}(\mathbf{A})) = \{0\}$ and all derivations of $\mathbf{A}$ are inner, i.e., $\mathrm{Der}(\mathbf{A}) = \mathrm{IDer}(\mathbf{A})$.

We define the ideals $\mathbf{A}^{(1)} = \mathbf{A} = \mathbf{A}^1$, $\mathbf{A}^{(i)} = [\mathbf{A}^{(i-1)}, \mathbf{A}^{(i-1)}]$ and $\mathbf{A}^i = [\mathbf{A}, \mathbf{A}^{i-1}]$ for $i \in \mathbb{Z}_{\geq 2}$. A Leibniz algebra $\mathbf{A}$ is said to be *solvable* (resp. *nilpotent*) if $\mathbf{A}^{(m)} = \{0\}$ (resp. $\mathbf{A}^m = \{0\}$) for some positive integer m. The *maximal solvable* (resp. *nilpotent*) ideal of $\mathbf{A}$ is called the *radical* (resp. *nilradical*) denoted by $\mathrm{rad}(\mathbf{A})$ (resp. $\mathrm{nilrad}(\mathbf{A})$). A Leibniz algebra $\mathbf{A}$ is called *simple* if its ideals are only $\{0\}$, $\mathrm{Leib}(\mathbf{A})$, $\mathbf{A}$ and $[\mathbf{A}, \mathbf{A}] \neq \mathrm{Leib}(\mathbf{A})$. A Leibniz algebra $\mathbf{A}$ is *semisimple* if $\mathrm{rad}(\mathbf{A}) = \mathrm{Leib}(\mathbf{A})$.

A *holomorph* of the Leibniz algebra $\mathbf{A}$ is defined to be the vector space $\mathrm{hol}(\mathbf{A}) := \mathbf{A} \oplus \mathrm{Der}(\mathbf{A})$, with the bracket defined by $[x + \delta_1, y + \delta_2] = [x,y] + \delta_1(y) + [L_x, \delta_2] + [\delta_1, \delta_2]$ for all $x, y \in \mathbf{A}$ and $\delta_1, \delta_2 \in \mathrm{Der}(\mathbf{A})$. For two subspaces M and N of $\mathrm{hol}(\mathbf{A})$, the *left centralizer* of M in N is defined to be $Z^{\ell}_N(M) = \{x \in N \mid [x,M] = 0\}$.

A derivation $d \in \mathrm{Der}(\mathbf{A})$ is called a *central derivation* if $\mathrm{im}(d) \subseteq Z(\mathbf{A})$. We denote $\mathrm{CDer}(\mathbf{A})$ to be the set of all central derivations of $\mathbf{A}$. The *centroid* $C(\mathbf{A})$ of $\mathbf{A}$ is the set of all linear maps $D : \mathbf{A} \to \mathbf{A}$ such that $D[x,y] = [D(x),y] = [x,D(y)]$ for all $x, y \in \mathbf{A}$. We denote $\mathrm{Aut}(\mathbf{A})$ to be the automorphism group of $\mathbf{A}$. Let G be a subgroup of $\mathrm{Aut}(\mathbf{A})$. A linear map $D : \mathbf{A} \to \mathbf{A}$ is called a *G-derivation* of $\mathbf{A}$ if there exist two automorphisms $\sigma, \tau \in G$ such that $D[x,y] = [D(x), \sigma(y)] + [\tau(x),D(y)]$ for all $x, y \in \mathbf{A}$. In this case, σ and τ are called *associated* automorphisms of D.

We denote $\mathrm{Der}_G(\mathbf{A})$ to be the set of all G-derivations of $\mathbf{A}$. Given two elements $\sigma, \tau \in G$, we denote $\mathrm{Der}_{\sigma,\tau}(\mathbf{A})$ to be the set of all G-derivations associated to σ and τ in particular, $\mathrm{Der}_{id,id}(\mathbf{A}) = \mathrm{Der}(\mathbf{A})$. For simplicity of notation, we denote $\mathrm{Der}_{\sigma,id}((\mathbf{A})$ as $\mathrm{Der}_{\sigma}(\mathbf{A})$. For a subalgebra M of $\mathbf{A}$ and an automorphism of $\mathbf{A}$ such that $\sigma(M) \subseteq M$, we denote $\mathrm{Der}_{\sigma,M}(\mathbf{A})$ to be the set of all σ-derivations of $\mathbf{A}$ which stabilizes M, i.e., $\mathrm{Der}_{\sigma,M}(\mathbf{A}) = \{D \in \mathrm{Der}_{\sigma}(\mathbf{A}) \mid D(M) \subseteq M\}$.

.

VITA

NAME	Sutida Patlertsin
DATE OF BIRTH	20 December 1991
PLACE OF BIRTH	Bangkok, Thailand
INSTITUTIONS ATTENDED	Bachelor of Science degree in Mathematics at Srinakharinwirot University, Master of Science degree in Mathematics at Chulalongkorn University
PUBLICATION	Patlertsin S, Pongprasert S, Rungratgasame T. On inner derivations of Leibniz algebras. Mathematics. 2024;12:1152(1-9).